3, 5, 7, . . . $(2n+1)$-bodies choreographies on the Lemniscate. Superintegrability

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In 2003 Fujiwara et al. discovered a remarkable 3-body choreography on the *algebraic* Lemniscate by Bernoulli for which the potential is found explicitly and depends on relative distances only

J. Phys. A: Math. Gen. 36 (2003)

$$(x^2+y^2)^2 = c^2(x^2-y^2)$$

The goal is to show that it is also possible to have choreographies with 5, 7 and "possibly" with any odd number of bodies on the algebraic Lemniscate. All choreographies are *superintegrable*.

DEFINITION

Choreographic motion of N identical bodies is a periodic motion on a closed orbit, chasing each other on the orbit with equal time-spacing.

Choreographic motions in Newtonian gravity

- 1993, Figure-eight three-body Newtonian choreographic numerical solution, C Moore (1993) Phys.Rev.Lett. 70, 36759
- 2000, Rigorous proof, A Chenciner and R Montgomery (2000) Ann.Math. 152, 881901
- 2001-present, Many remarkable choreographic N-body numerical solutions on the plane are found (see C Simó et al)

Remarkable discovery (2003):

- Choreographic three bodies on the algebraic lemniscate by

  Toshiaki Fujiwara, Hiroshi Fukuda and Hiroshi Ozaki

  J. Phys. A: Math. Gen. 36 (2003) 2791-2800

3-body choreographic motion in (non)-Newtonian gravity: at 2016

- Newtonian 3-body in $2D$

$$U = \sum \log r_{ij}$$

-

$$U = -\alpha \sum \frac{1}{r_{ij}^{\alpha}} \quad , \quad 2 \geq \alpha > -2$$

-

$$U = -\sum \frac{1}{r_{ij}^{6}}$$

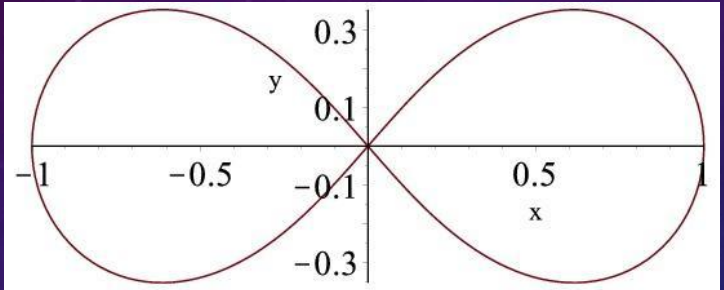
- Lennard-Jones potential

$$U = \sum \left(\frac{1}{r_{ij}^{12}} - \frac{1}{r_{ij}^{6}} \right)$$

Lemniscate, Parametrization

$$(x^2 + y^2)^2 = c^2(x^2 - y^2)$$

for $c = 1$



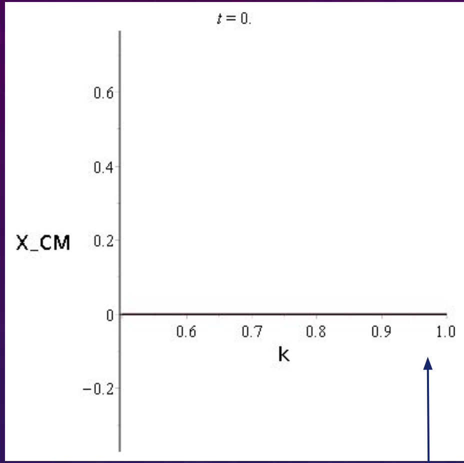
$$x(t) = \frac{c\,\mathrm{sn}(t,k)}{1+\mathrm{cn}^2(t,k)} \quad , \quad y(t) = \frac{c\,\mathrm{sn}(t,k)\,\mathrm{cn}(t,k)}{1+\mathrm{cn}^2(t,k)}$$

$\mathrm{sn}(t,k), \mathrm{cn}(t,k)$: Jacobi elliptic functions, $k \in [0,1]$ elliptic modulus

$$\dot{x}^2 + \dot{y}^2 + (k^2 - 1/2)(x^2 + y^2) = c^2/2$$

Period: $T = 4\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ ← complete elliptic integral, c-indept

3-Bodies on the Lemniscate



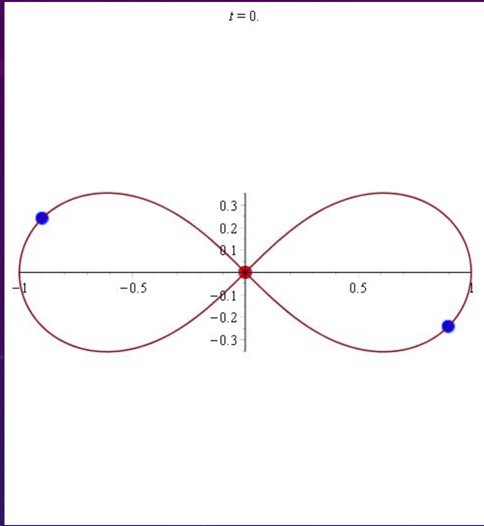
3-bodies choreography on the Lemniscate:

$$x_1(t) = x(t) \quad , \quad y_1(t) = y(t)$$
$$x_2(t) = x(t + T/3) \quad , \quad y_2(t) = y(t + T/3)$$
$$x_3(t) = x(t - T/3) \quad , \quad y_3(t) = y(t - T/3)$$

assuming $m_i = 1, \quad {\scriptstyle i=1\ldots3}$

$$\mathbf{X}_{\mathrm{CM}}(t) = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = 0$$

3-Bodies on the Lemniscate



If and only if

$$k_0^2 = \frac{2+\sqrt{3}}{4} = \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^2$$

k_0 and period $T = 4K(k_0)$ satisfy the equation

$$\mathrm{sn}\left(\frac{T(k_0)}{12}, k_0\right) = \sqrt{3} - 1$$

Fujiwara et al. 2003

then

it implies

$$x_1 + x_2 + x_3 = 0 \quad , \quad y_1 + y_2 + y_3 = 0$$

hence

$$x(t) + x(t - \frac{T}{3}) + x(t + \frac{T}{3}) = 0 \, , \; y(t) + y(t - \frac{T}{3}) + y(t + \frac{T}{3}) = 0$$

$$\frac{\mathrm{sn}(t, k_0)}{1 + \mathrm{cn}^2(t, k_0)} + \frac{\mathrm{sn}(t - \frac{4K}{3}, k_0)}{1 + \mathrm{cn}^2(t - \frac{4K}{3}, k_0)} + \frac{\mathrm{sn}(t + \frac{4K}{3}, k_0)}{1 + \mathrm{cn}^2(t + \frac{4K}{3}, k_0)} = 0$$

$$\frac{\mathrm{sn}(t, k_0)\,\mathrm{cn}(t, k_0)}{1 + \mathrm{cn}^2(t, k_0)} + \frac{\mathrm{sn}(t - \frac{4K}{3}, k_0)\,\mathrm{cn}(t - \frac{4K}{3}, k_0)}{1 + \mathrm{cn}^2(t - \frac{4K}{3}, k_0)} +$$

$$\frac{\mathrm{sn}(t + \frac{4K}{3}, k_0)\,\mathrm{cn}(t + \frac{4K}{3}, k_0)}{1 + \mathrm{cn}^2(t + \frac{4K}{3}, k_0)} = 0$$

Initial Conditions

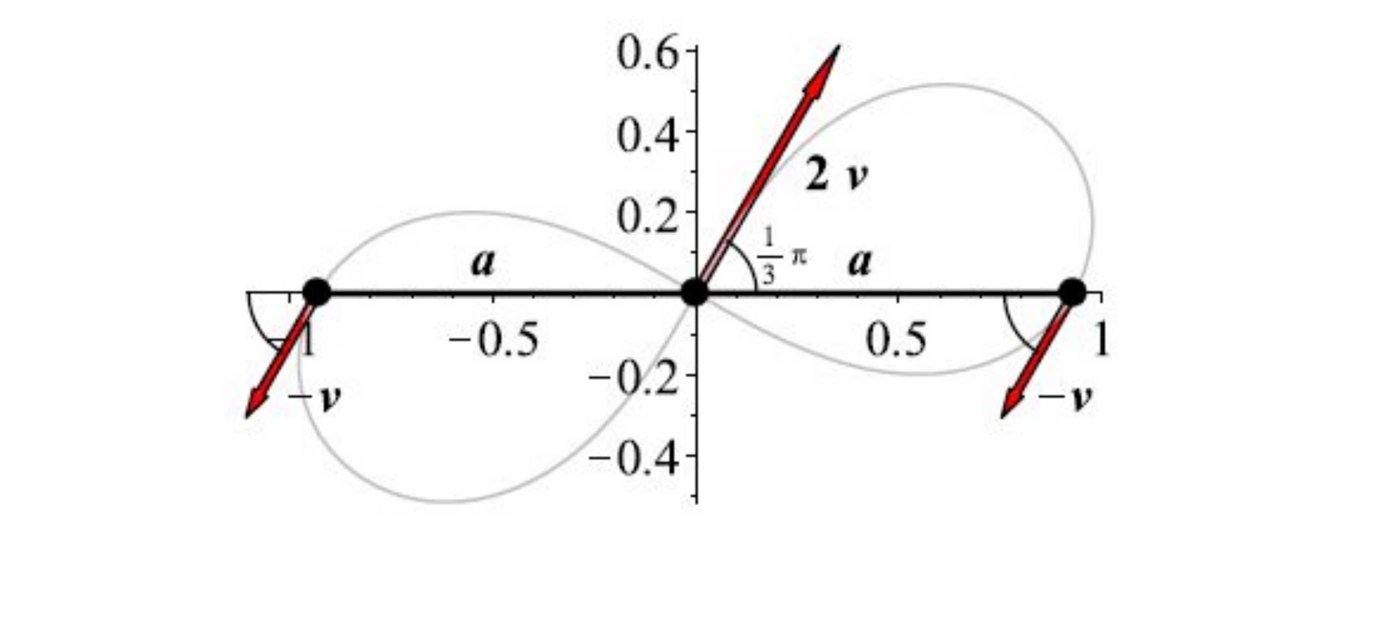
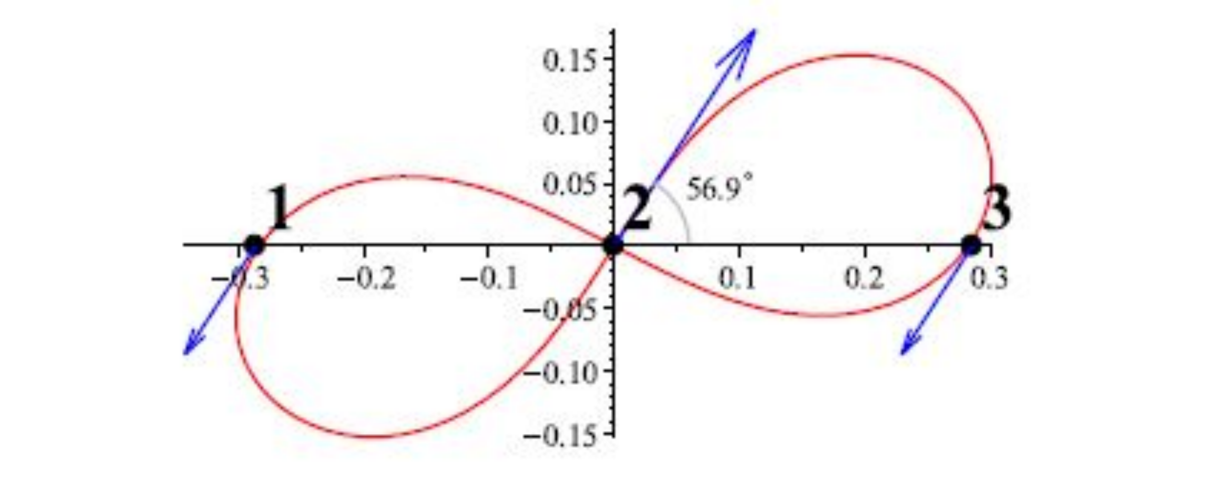


Figure: Initial conditions: $a^2 = \frac{\sqrt{3}}{2}c^2$, $|2\mathbf{v}| = \frac{c}{\sqrt{2}}$

Initial Conditions. Figure-8 Simo (Newtonian Gravity)

Carles Simo, Contemporary Mathematics, 292 (2002)
Dynamical properties of the figure eight solution of the three body problem



The initial conditions are: $x_1 = -x_3 = -0.27628526570712499492$, $y_1 = -y_3 = 0.07403041382684230254$3, $\dot{x}_1 = \dot{x}_3 = -\frac{1}{2}\dot{x}_2$, $\dot{x}_2 = 1.02159379446156597$64, $\dot{y}_1 = \dot{y}_3 = -\frac{1}{2}\dot{y}_2$, $\dot{y}_2 = 0.917017674936815908$62. Velocities are scaled in the figure. The normalizations used are: sum of the masses equal to 1, total energy equal to $-1/2$, center of mass kept fixed at the origin.

Comparison

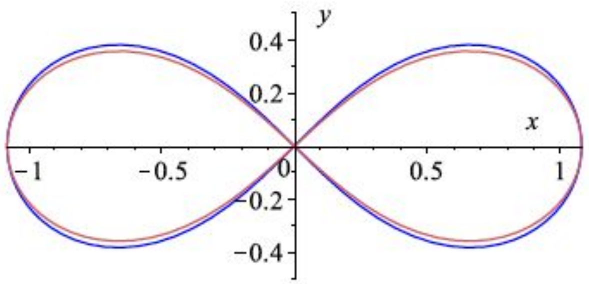


Figure: Simo Figure-eight for $m_1 = m_2 = m_3 = 1$ (red) vs scaled algebraic Lemniscate for $c = 1.08101708150691$ (blue).

Initial conditions for Simo's Figure 8 (Euler line): $x_1 = -x_2 = 0.97000436$, $x_3 = 0$,

$y_1 = -y_2 = -0.24308753$, $y_3 = 0$, $\dot{x}_3 = -2\dot{x}_1 = -2\dot{x}_2 = -0.93240737$, $\dot{y}_3 = -2\dot{y}_l = -2\dot{y}_2 = -0.86473146$.

Evolution

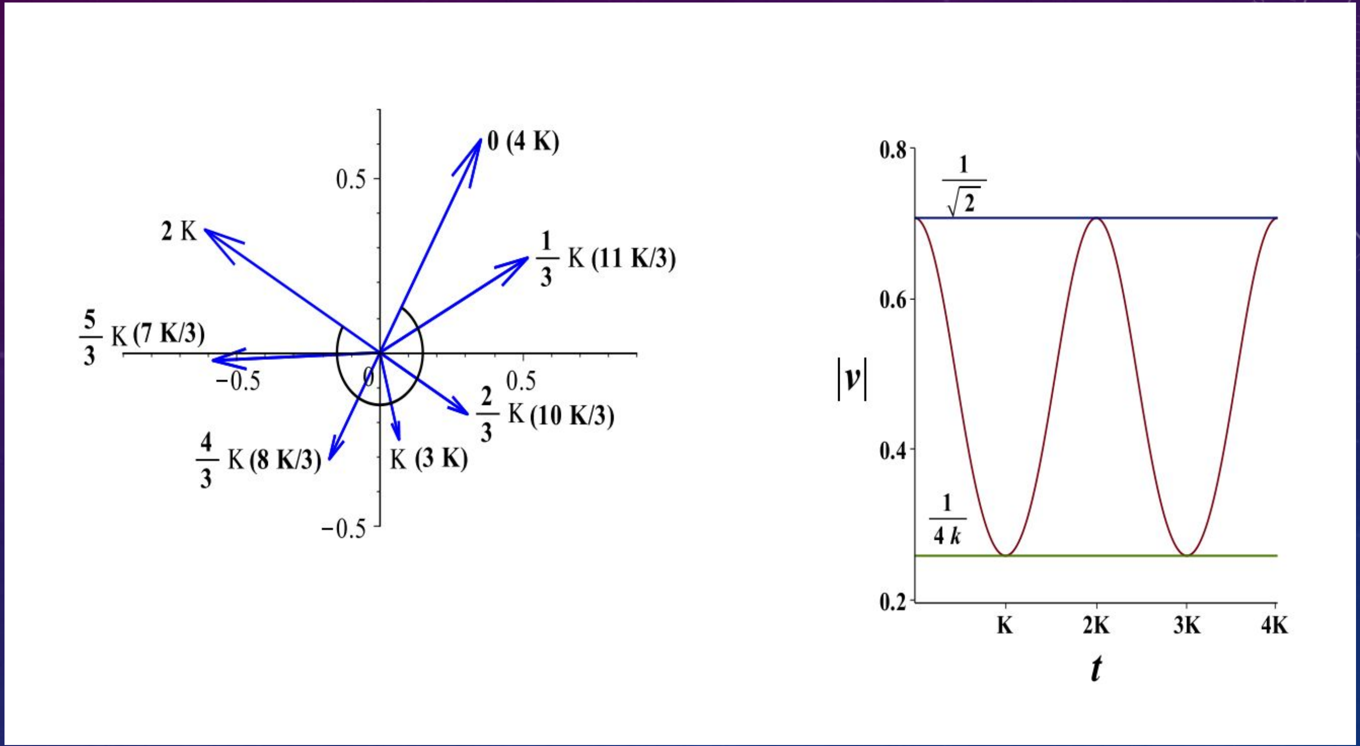


Figure: Evolution of the velocity of the body starting at the origin.

Evolution of the relative distance r_{12}

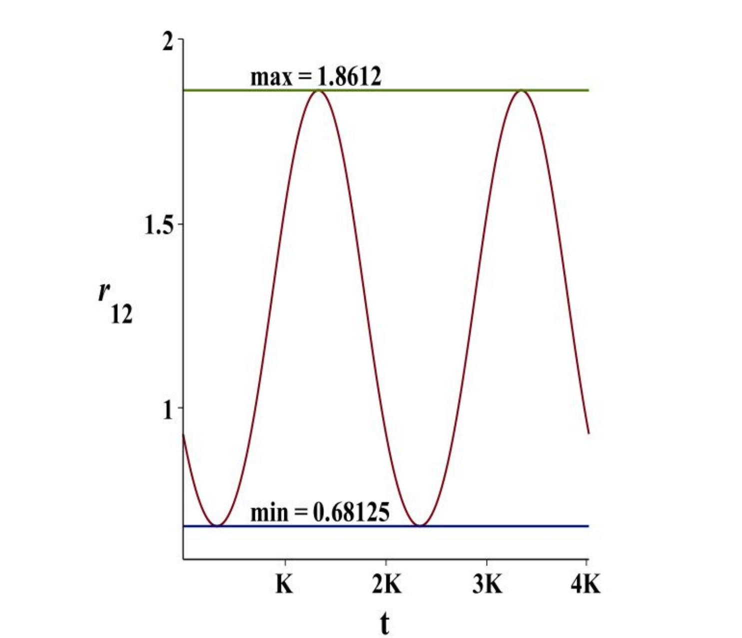


Figure:

for $t = 0$, $\quad r_{12} = \sqrt{\frac{\sqrt{3}}{2}}$

for $t = K/3$, $\quad r_{12} = r_{12}^{min} = 0.68125003863321328035$

for $t = 4K/3$, $\quad r_{12} = r_{12}^{max} = 1.8612097182041991978$

There exists **only** two particular, velocity-independent constants of motion for $k = k_0$:

1. $I_1 \equiv r_{12}^2\, r_{13}^2\, r_{23}^2 = \frac{3\sqrt{3}}{2}$
2. $I_2 \equiv r_{12}^2 + r_{13}^2 + r_{23}^2 = 3\sqrt{3}$

both $I_{1,2}$ are S_3-invariant, here

$$r_{ij} = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)^2}$$

Direct calculation of kinetic energy shows that

Kinetic Energy is **constant of motion***!*

3. $\mathcal{T} = \frac{1}{2}\left(\mathbf{v}_1^2 + \mathbf{v}_2^2 + \mathbf{v}_3^2\right) = \frac{3}{8}$

- It implies, that the potential energy $\mathcal{V}$ is a constant of motion as well(!)

$$\mathcal{V} = \mathcal{V}(I_1, I_2)$$

- Assuming pairwise interaction only, it leads to

$$\mathcal{V} = \alpha \log I_1 - \beta I_2$$

List of conserved quantities, global and particular

1. $\sum \mathbf{x}_i \times \mathbf{v}_i = \mathbf{x}_{13} \times \mathbf{v}_1 + \mathbf{x}_{23} \times \mathbf{v}_2 = 0$ Angular Momentum
2. $E = \mathcal{T} + \mathcal{V}$ Total Energy (dependable) $= \frac{1}{4} \log(\frac{3\sqrt{3}}{2}) \simeq 0.23869$
3. $I_1 = r_{12}^2\, r_{13}^2\, r_{23}^2 = \frac{3\sqrt{3}}{2}$
4. $I_2 = I_{HR}^{(3)} = 3 \sum \mathbf{x}_i^2 = \sum_{i<j} r_{ij}^2 = 3\sqrt{3}$ Moment of Inertia

   or hyper-radius squared, $r_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^2 \equiv \rho_{ij}$
5. $\tilde{\mathcal{T}} = \mathbf{v}_1^2\, \mathbf{v}_2^2\, \mathbf{v}_3^2 = \frac{1}{128}$
6. $\mathcal{T} = \frac{1}{2} \sum \mathbf{v}_i^2 = \frac{3}{8}$ Kinetic Energy

List of conserved quantities, global and particular

7. $J_1 = \mathbf{v}_1^2 + \frac{1}{9}(k_0^2 - \frac{1}{2})\,(2\,r_{12}^2 + 2\,r_{13}^2 - r_{23}^2) = \frac{1}{2}$

8. $J_2 = \mathbf{v}_2^2 + \frac{1}{9}(k_0^2 - \frac{1}{2})\,(2\,r_{12}^2 - r_{13}^2 + 2\,r_{23}^2) = \frac{1}{2}$

9. $J_3 = \mathbf{v}_3^2 + \frac{1}{9}(k_0^2 - \frac{1}{2})\,(-\,r_{12}^2 + 2\,r_{13}^2 + 2\,r_{23}^2) = \frac{1}{2}$
(dependable)

Constraint:

$$\sum^{3} J_i(k_0) = 2\,\mathcal{T} + \frac{1}{3}(k_0^2 - \frac{1}{2})I_2$$

7 Independent conserved quantities

3-bodies on the Lemniscate

- Lemniscate for 3-body is a particularly maximally superintegrable trajectory, *i.e.*

$$\{H, I_j\}_{\mathrm{PB}}\,|_{\mathrm{trajectory}} = 0\ , \qquad j = 1..6$$

7 constants of motion after removing CM
(2 global and 5 particular)
All are polynomial in coordinates and momenta!

T-conjecture (A. Turbiner, 2013):

Any closed periodic trajectory is particularly (maximally) superintegrable

The 3-body choreography satisfies four Newton equations for relative motion

$$\frac{d^2}{dt^2}\mathbf{x}(t) = -\nabla_{\mathbf{x}}\mathcal{V}$$

in three independent variables r_{ij}.
Consistency conditions for those equations to satisfy leads to $\alpha = 1/4$ and $\beta = \sqrt{3}/24$ **unambiguosly**

$$\mathcal{V} = \frac{1}{4}\log I_1 - \frac{\sqrt{3}}{24}I_2 = \sum_{i<j}\left\{\frac{1}{4}\log r_{ij}^2 - \frac{\sqrt{3}}{24}r_{ij}^2\right\} = \mathcal{V}(r_{12}) + \mathcal{V}(r_{13}) + \mathcal{V}(r_{23})$$

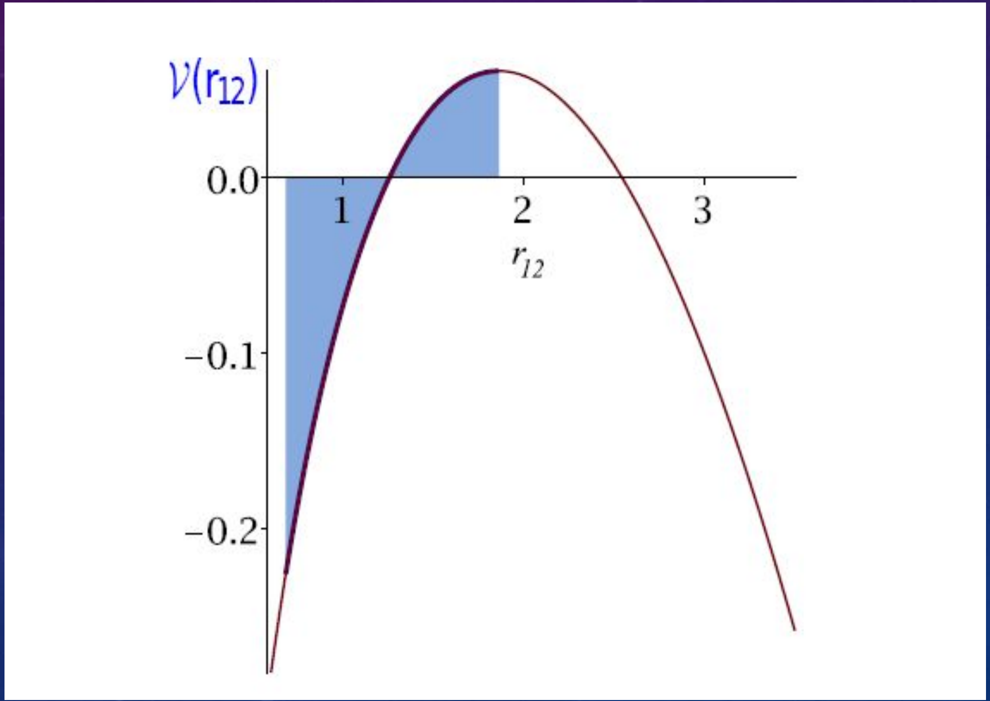
this work

Fujiwara *et al.* 2003

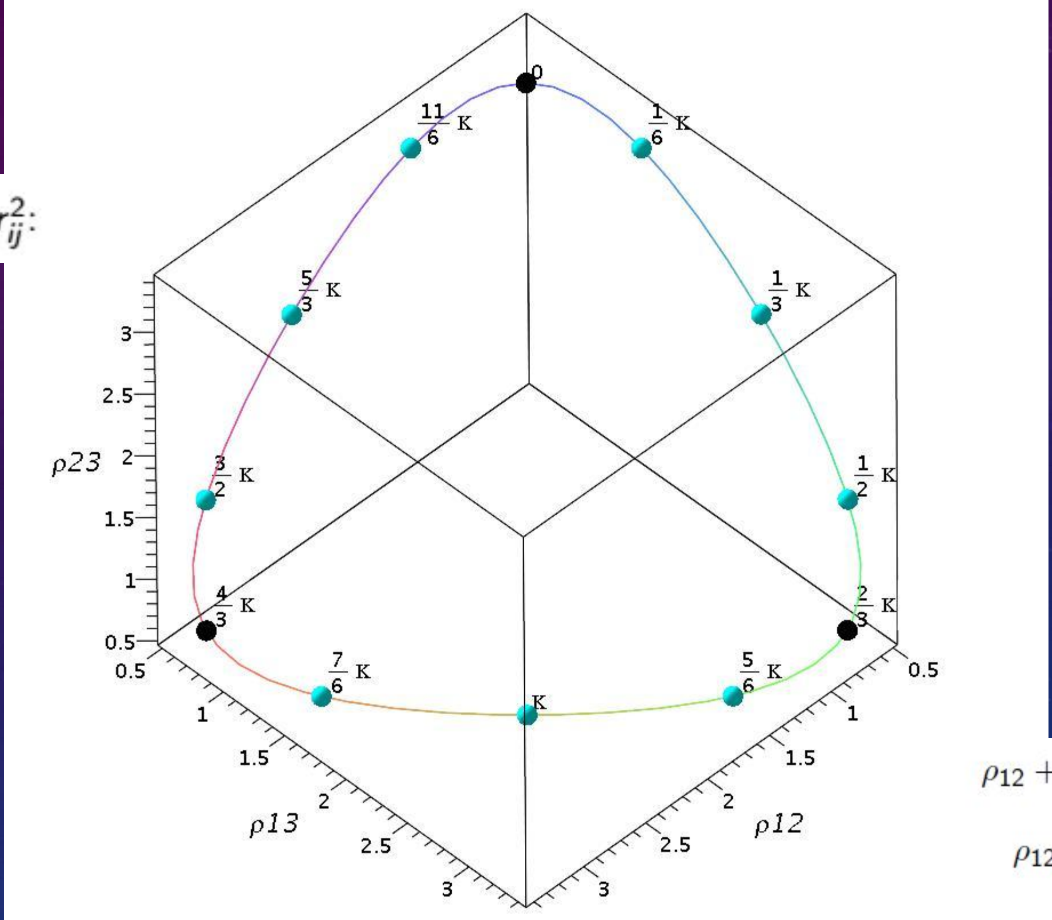
$$\mathcal{V}(r_{ij}) \equiv \left\{\frac{1}{4}\log r_{ij}^2 - \frac{\sqrt{3}}{24}r_{ij}^2\right\}$$

Pairwise potential $\mathcal{V}(r_{12})$

$$\mathcal{V}(r_{12}) \equiv \left\{ \frac{1}{4} \log r_{12}^2 - \frac{\sqrt{3}}{24} r_{12}^2 \right\}$$



In coordinates $\rho_{ij} = r_{ij}^2$:

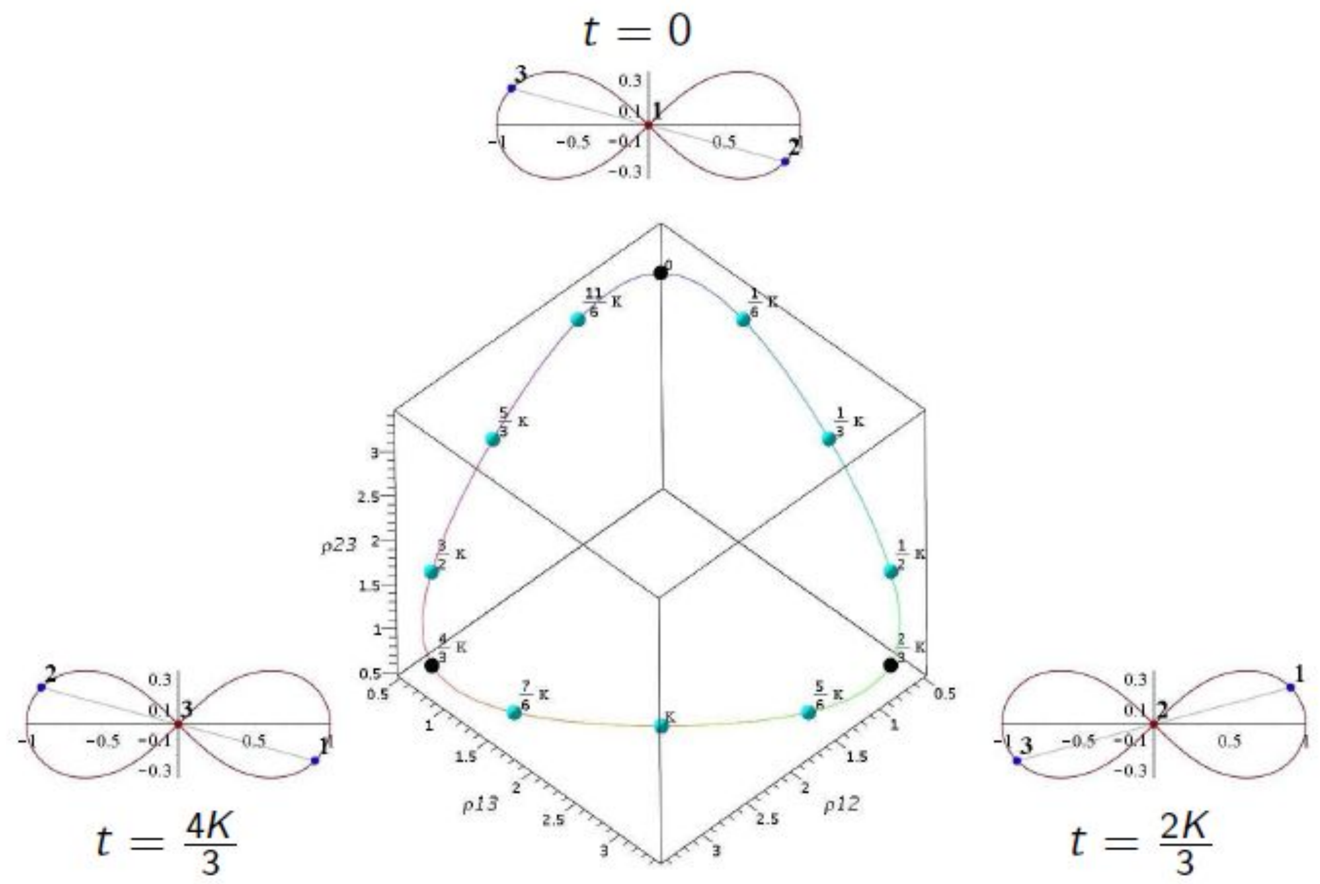


$$\rho_{12} + \rho_{13} + \rho_{23} = 3\sqrt{3}$$

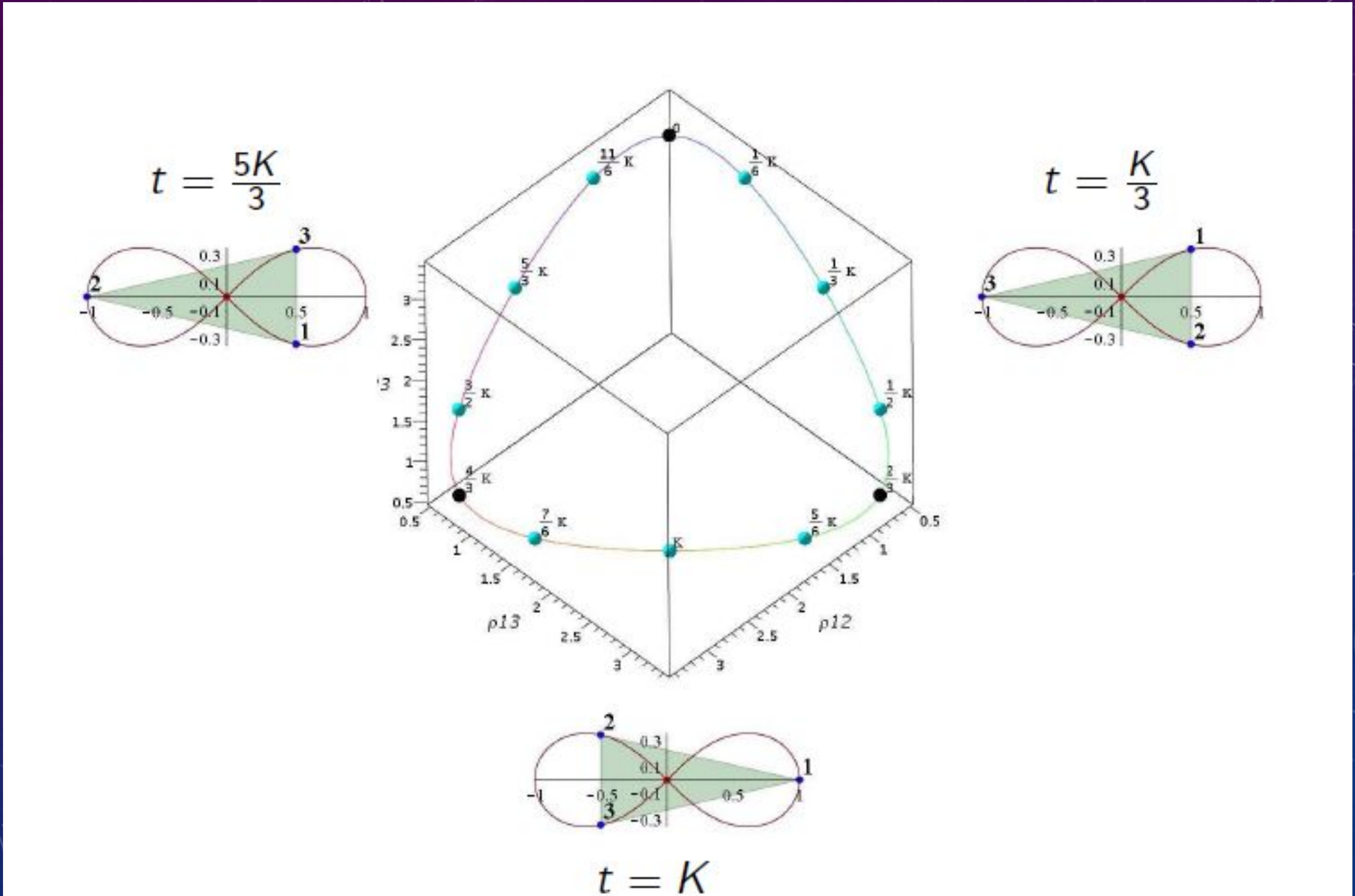
$$\rho_{12}\,\rho_{13}\,\rho_{23} = \frac{3\sqrt{3}}{2}$$

planar curve

(in $\mathbf{v}^2$-space it is a similar planar curve)



$t = 0$
$t = \frac{4K}{3}$
$t = \frac{2K}{3}$
ρ23
ρ13
ρ12
0
$\frac{1}{6}$ K
$\frac{1}{3}$ K
$\frac{1}{2}$ K
$\frac{2}{3}$ K
$\frac{5}{6}$ K
K
$\frac{7}{6}$ K
$\frac{4}{3}$ K
$\frac{3}{2}$ K
$\frac{5}{3}$ K
$\frac{11}{6}$ K

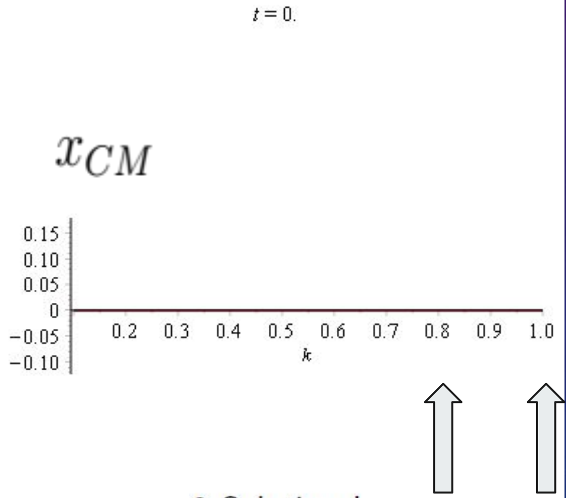


$t = \frac{5K}{3}$
$t = \frac{K}{3}$
$t = K$
ρ13
ρ12

5-Bodies on the Lemniscate

Center of Mass

$$\mathbf{X}_{\mathrm{CM}}(t) = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_5 = 0 \quad \Rightarrow \quad \text{2 Solutions!}$$



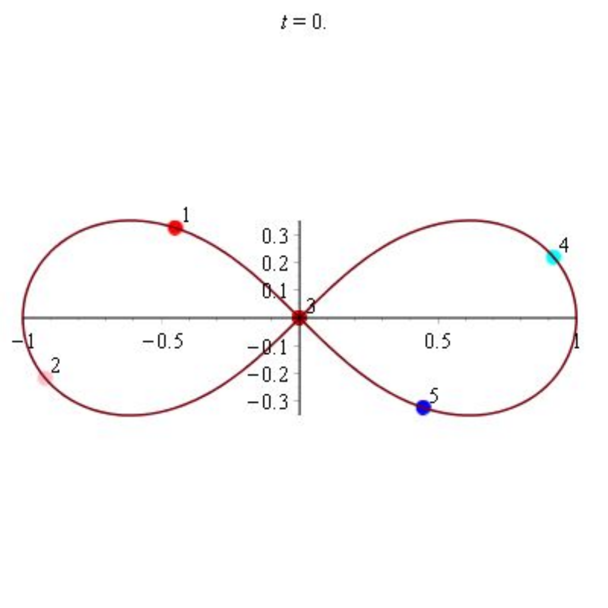
2 Solutions!

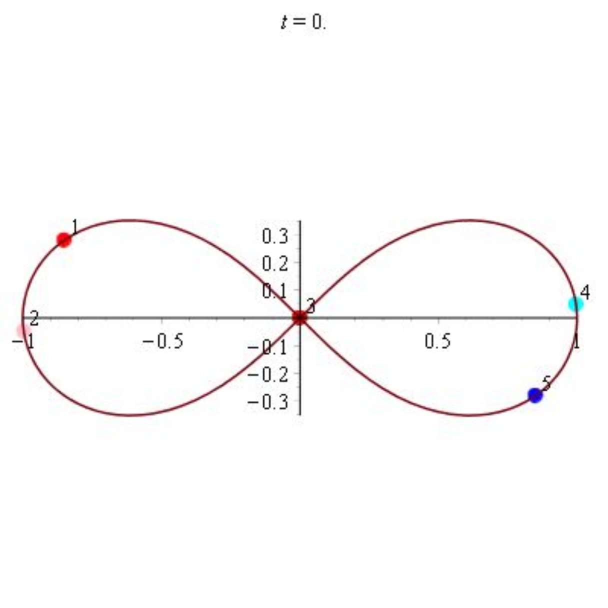
$$\begin{aligned}
x_1(t) &= x(t-2T/5)\,, & y_1(t) &= y(t-2T/5)\,,\\
x_2(t) &= x(t-T/5)\,, & y_2(t) &= y(t-T/5)\,,\\
x_3(t) &= x(t)\,, & y_3(t) &= y(t)\,,\\
x_4(t) &= x(t+T/5)\,, & y_4(t) &= y(t+T/5)\,,\\
x_5(t) &= x(t+2T/5)\,, & y_5(t) &= y(t+2T/5)\,,
\end{aligned}$$

$$k^2 = \begin{cases} 0.65366041395477321345 = k_1^2 \\ 0.99764373603161323509 = k_2^2 \end{cases}$$

Fujiwara *et al.* 2004

5-bodies choreographies on the Lemniscate





k_1=0.8084926802110042687

k_2=0.99882117319949382173

5-Bodies on the Lemniscate

Equations to find the periods and k's

$$\mathrm{dn}\left(\frac{7K(k_1)}{5}, k_1\right) - \frac{1}{\sqrt{2}} = 0,$$

$$\mathrm{dn}\left(\frac{9K(k_2)}{5}, k_2\right) - \frac{1}{\sqrt{2}} = 0,$$

Fujiwara et al. (2004)

k_1=0.80849268021100426871

k_2=0.99882117319994938217

Initial conditions

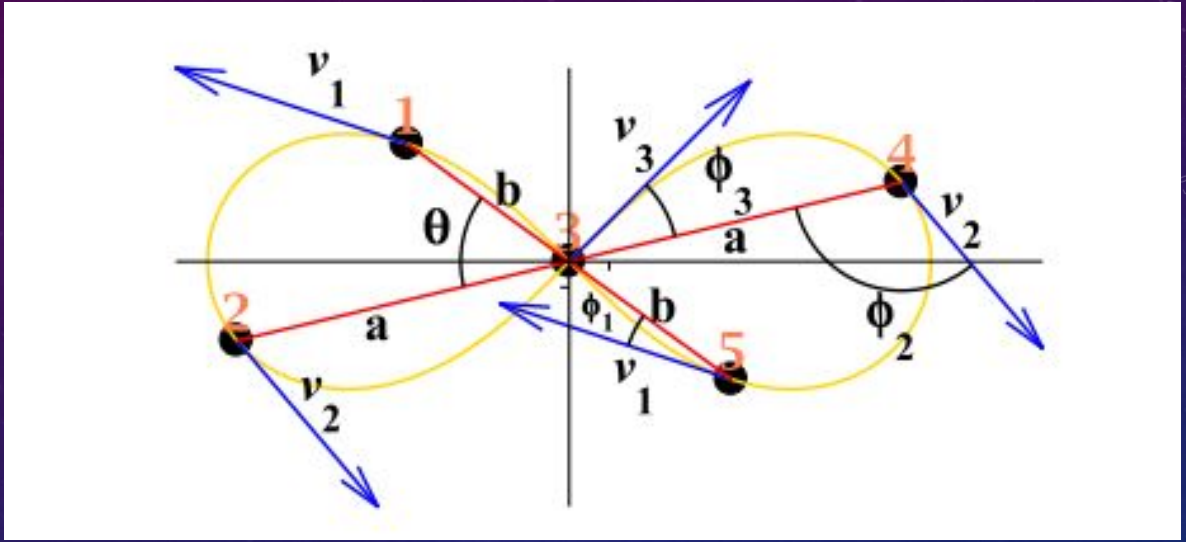


Figure: Initial conditions for k_1:
$a = 0.944572650815175422 56$, $b = 0.5543652477469582750 5$
Angle between the two Eulerian lines is $\theta = 49.47446127331187448 6°$.
$\mathbf{v}_1 = 0.67288702942358420449$, $\mathbf{v}_2 = .602413056580735275 92$
$\phi_1 = 17.897845093247497946$, $\phi_2 = 116.84676763987124692°$
$\phi_3 = 31.576616180064376539°$

5-Bodies on the Lemniscate

N-body Choreography on the Lemniscate

(Developments and Applications of Dynamical Systems Theory)
Toshiaki Fujiwara, Hiroshi Fukuda, Hiroshi Ozaki
(2004)

… We also investigate the same parameterization with different modulus for N-body system with N=5,7, 9, … on the lemniscate. We show that it conserves the center of mass, the angular momentum and the moment of inertia, but that it may not satisfy equation of motion under any interaction potential, which means unfortunately it is not a N-body choreography.

…

… Or at least, it is very difficult to find such equation of motion.

there exist three particular, velocity-independent constants of motion different for each k:

1. $$I_1^{(5)} = \begin{cases} r_{12}^2 r_{23}^2 r_{34}^2 r_{45}^2 r_{15}^2 = 0.2636217830340870711\,0 \ (k_1) \\ r_{13}^2 r_{35}^2 r_{25}^2 r_{24}^2 r_{14}^2 = 30.760801541637359790 \ (k_2) \end{cases}$$

2. $$I_2^{(5)} = \begin{cases} r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{45}^2 + r_{15}^2 = 4.0517817845468308414 \ (k_1) \\ r_{13}^2 + r_{35}^2 + r_{25}^2 + r_{24}^2 + r_{14}^2 = 12.515257719766335417 \ (k_2) \end{cases}$$

3. $$I_{\mathrm{HR}}^{(5)} = 5\sum^5 \mathbf{x}_i^2 = \sum_{i<j}^5 r_{ij}^2 = \begin{cases} 11.995383205775537457 \ (k_1) \\ 17.975523091392961251 \ (k_2) \end{cases}$$

here

$$r_{ij} = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)^2}$$

Direct calculation shows that

Kinetic Energy is **constant of motion**!

4. $\mathcal{T} = \frac{1}{2}\sum^5 \mathbf{v}_i^2 = \begin{cases} 1.0656784451054396\ (k_1) \\ 0.3554593531676729\ (k_2) \end{cases}$

- It implies, that the potential energy $\mathcal{V}$ is a constant of motion as well(!)

$$\mathcal{V} = \mathcal{V}(I_1^{(5)}, I_2^{(5)}, I_{HR})$$

- Assuming pairwise interaction only, it leads to

$$\mathcal{V} = \alpha\ \log I_1^{(5)} + a\ I_2^{(5)} - \beta\ I_{\mathrm{HR}}^{(5)}$$

$$a = 0 \text{ (see below)}$$

List of conserved quantities, global and particular

1. $\sum \mathbf{x}_i \times \mathbf{v}_i = 0$ Angular Momentum

2. $E = \mathcal{T} + \mathcal{V}$ Total Energy (dependable) $\begin{cases} 0.54804692944384581934 \ (k_1) \\ 0.31747900688996754830 \ (k_2) \end{cases}$

3. $I_1^{(5)} = \begin{cases} r_{12}^2 r_{23}^2 r_{34}^2 r_{45}^2 r_{15}^2 = 0.26362178303408707110 \ (k_1) \\ r_{13}^2 r_{35}^2 r_{25}^2 r_{24}^2 r_{14}^2 = 30.760801541637359790 \ (k_2) \end{cases}$

4. $I_a^{(5)} = r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{45}^2 + r_{15}^2 = \begin{cases} 4.0517817845468308414 \ (k_1) \\ 5.4602653716266258340 \ (k_2) \end{cases}$

5. $I_b^{(5)} = r_{13}^2 + r_{35}^2 + r_{25}^2 + r_{24}^2 + r_{14}^2 = \begin{cases} 7.9436014212287066156 \ (k_1) \\ 12.515257719766335417 \ (k_2) \end{cases}$

6. $I_{\mathrm{HR}}^{(5)} = 5 \sum^{5} \mathbf{x}_i^2 = \sum_{i<j}^{5} r_{ij}^2 = I_a^{(5)} + I_b^{(5)} = \begin{cases} 11.995383205775537457 \ (k_1) \\ 17.975523091392961251 \ (k_2) \end{cases}$

(dependable)

List of conserved quantities, global and particular

6. $\sum \rho_i^{-2} = \frac{9}{5} I_{\mathrm{HR}}^{(5)} = \begin{cases} 21.591689770395967423 \ (k_1) \\ 32.355941564507330250 \ (k_2) \end{cases}$
   (dependable)

7. $\tilde{\mathcal{T}} = \mathbf{v}_1^2\mathbf{v}_2^2\mathbf{v}_3^2\mathbf{v}_4^2\mathbf{v}_5^2 = \begin{cases} 0.01349945192046077 \ (k_1) \\ 1.121985660295881 \times 10^{-7} \ (k_2) \end{cases}$

8. $\mathcal{T} = \frac{1}{2}\sum^5 \mathbf{v}_i^2 = \begin{cases} 1.0656784451054396 \ (k_1) \\ 0.3554593531676 6729 \ (k_2) \end{cases}$

9. $J_1(k_{1,2}) = \mathbf{v}_1^2 + \frac{1}{25}\left(k_{1,2}^2 - \frac{1}{2}\right)\Big(b_1^{(1)} r_{12}^2 + b_2^{(1)} r_{13}^2 + b_3^{(1)} r_{14}^2 + b_4^{(1)} r_{15}^2 + b_5^{(1)} r_{23}^2$
   $+ b_6^{(1)} r_{24}^2 + b_7^{(1)} r_{25}^2 + b_8^{(1)} r_{34}^2 + b_9^{(1)} r_{35}^2 + b_{10}^{(1)} r_{45}^2\Big) = \frac{1}{2}$

$\vdots$

13. $J_5(k_{1,2}) = \mathbf{v}_5^2 + \frac{1}{25}\left(k_{1,2}^2 - \frac{1}{2}\right)\Big(b_1^{(5)} r_{12}^2 + b_2^{(5)} r_{13}^2 + b_3^{(5)} r_{14}^2 + b_4^{(5)} r_{15}^2 + b_5^{(5)} r_{23}^2$
   $+ b_6^{(5)} r_{24}^2 + b_7^{(5)} r_{25}^2 + b_8^{(5)} r_{34}^2 + b_9^{(5)} r_{35}^2 + b_{10}^{(5)} r_{45}^2\Big) = \frac{1}{2}$ (dependable)

Constraint: $\sum_{i=1}^{5} J_i(k_{1,2}) = 2\,\mathcal{T} + \frac{1}{5}(k_{1,2}^2 - \frac{1}{2}) I_{HR}^{(5)}$

where the coefficients (rounded to 5 d.d.) in J_i are:

k_1	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$\sum_{i=1}^{5} b_j^{(i)}$
$b_1^{(i)}$	-329.147524	-59.69546	-87.73055	-234.83703	716.41056	5
$b_2^{(i)}$	-20.059028	-90.14054	-5.51093	-33.82040	154.53089	5
$b_3^{(i)}$	-9.045432	-0.21873	-16.35914	-7.82114	38.44444	5
$b_4^{(i)}$	40.597330	52.33418	-37.13025	-14.49296	-36.30831	5
$b_5^{(i)}$	51.064166	-77.54409	78.29565	74.80036	-121.61610	5
$b_6^{(i)}$	-3.637906	11.41297	4.00945	25.18015	-31.96466	5
$b_7^{(i)}$	23.142507	90.30732	3.66870	78.77269	-190.89121	5
$b_8^{(i)}$	-47.726676	-418.37579	1.35333	-295.98006	765.72919	5
$b_9^{(i)}$	66.460587	11.71002	20.03987	59.13346	-152.34393	5
$b_{10}^{(i)}$	188.538681	472.85253	48.54940	247.21732	-952.15793	5

k_2	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$\sum_{i=1}^{5} b_j^{(i)}$
$b_1^{(i)}$	-11.52801	-15.16161	5.16178	-22.33573	48.86356	5
$b_2^{(i)}$	4.88928	3.32090	9.25792	2.94988	-15.41798	5
$b_3^{(i)}$	-6.71121	-6.67925	-7.56683	-5.90072	31.85802	5
$b_4^{(i)}$	4.28777	3.61262	3.31595	3.97331	-10.18965	5
$b_5^{(i)}$	-4.48027	-4.27050	2.96049	-3.25971	14.04999	5
$b_6^{(i)}$	12.85954	15.96479	-0.39468	12.94747	-36.37713	5
$b_7^{(i)}$	-1.09041	-3.71800	-0.76787	-2.63833	13.21461	5
$b_8^{(i)}$	-30.62669	-31.56751	16.17277	-25.15733	76.17876	5
$b_9^{(i)}$	19.78069	29.92687	-11.36654	25.72760	-59.06862	5
$b_{10}^{(i)}$	-9.33062	-25.11975	13.69066	-12.59512	38.35483	5

1. $$I_2^{(5)} = r_{13}^2 + ar_{34}^2 + br_{45}^2 + cr_{15}^2 = \begin{cases} -4.9893731063757473516 \ (k_1) \\ 4.5923978945218266705 \ (k_2) \end{cases}$$

$$\begin{array}{llll} a = -1, & b = -6.1175316692890807486, & c = -1 & (k_1) \\ a = 0.89783594223579568525, & b = 0.11755596103340095683, & c = 0.89783594223579419754 & (k_2) \end{array}$$

2. $$I_3^{(5)} = r_{12}^2 + ar_{14}^2 + r_{34}^2 + br_{45}^2 + cr_{15}^2 = \begin{cases} 3.8685701539367930365 \ (k_1) \\ 5.0629405424126776331 \ (k_2) \end{cases}$$

$$\begin{array}{llll} a = 0.19540670475985851441, & b = 1.1954067047598568846, & c = 0.1954067047598574790 & (k_1) \\ a = 1.2815912545378342469, & b = -0.15065869157918997387, & c = -0.15065869157918410114 & (k_2) \end{array}$$

3. $$I_4^{(5)} = r_{12}^2 + ar_{24}^2 + r_{45}^2 + br_{15}^2 = \begin{cases} 4.9893731063757975999 \ (k_1) \\ 5.1149633006284987259 \ (k_2) \end{cases}$$

$$\begin{array}{lll} a = -1, & b = 6.1175316692891318543 & (k_1) \\ a = 1.1137892269156327748, & b = 0.13093256295872669530 & (k_2) \end{array}$$

4. $$I_5^{(5)} = r_{12}^2 + ar_{34}^2 + br_{25}^2 + r_{15}^2 = \begin{cases} -0.93759132182895716222 \ (k_1) \\ 0.34530207099754785980 \ (k_2) \end{cases}$$

$$\begin{array}{lll} a = -5.1175316692891327860, & b = 1 & (k_1) \\ a = 0.86906743704145139444, & b = -1.1137892269159841664 & (k_2) \end{array}$$

Fourteen independent conserved quantities

One more constant of motion

5. $I_6^{(5)} = ar_{12}^2 + br_{14}^2 + cr_{24}^2 + r_{35}^2 \begin{cases} -3.8312045182313851648 \ (k_1) \\ 9.5199479346308316909 \ (k_2) \end{cases}$

$a = -5.9221249645292276533, \quad b = -0.19540670475985851441, \quad c = -0.19540670475985851441 \quad (k_1)$

$a = 1.15065869157918997387, \quad b = 1.15065869157918997387, \quad c = 1.15065869157918997387 \quad (k_2)$

Fifteen independent conserved quantities

5 bodies on the lemniscate

- The Lemniscate for 5-body is a particularly maximally superintegrable trajectory

(T-conjecture (2013) holds)

For 5-body case maximal particular superintegrability implies 15 constants of motion

we found 15 constants of motion!

5 Bodies on the Lemniscate: Potential

It was found (Lopez Vieyra, 2019)

$$\mathcal{V} = \alpha \log I_1^{(5)} - \beta I_{\mathrm{HR}}^{(5)}$$

$$\mathcal{V} = \begin{cases} \alpha_1 \{\log r_{12}^2 + \log r_{23}^2 + \log r_{34}^2 + \log r_{45}^2 + \log r_{15}^2\} \\ \alpha_2 \{\log r_{13}^2 + \log r_{35}^2 + \log r_{25}^2 + \log r_{24}^2 + \log r_{14}^2\} \end{cases} - \beta_{1,2} \sum_{i<j} r_{ij}^2 ,$$

Potential:

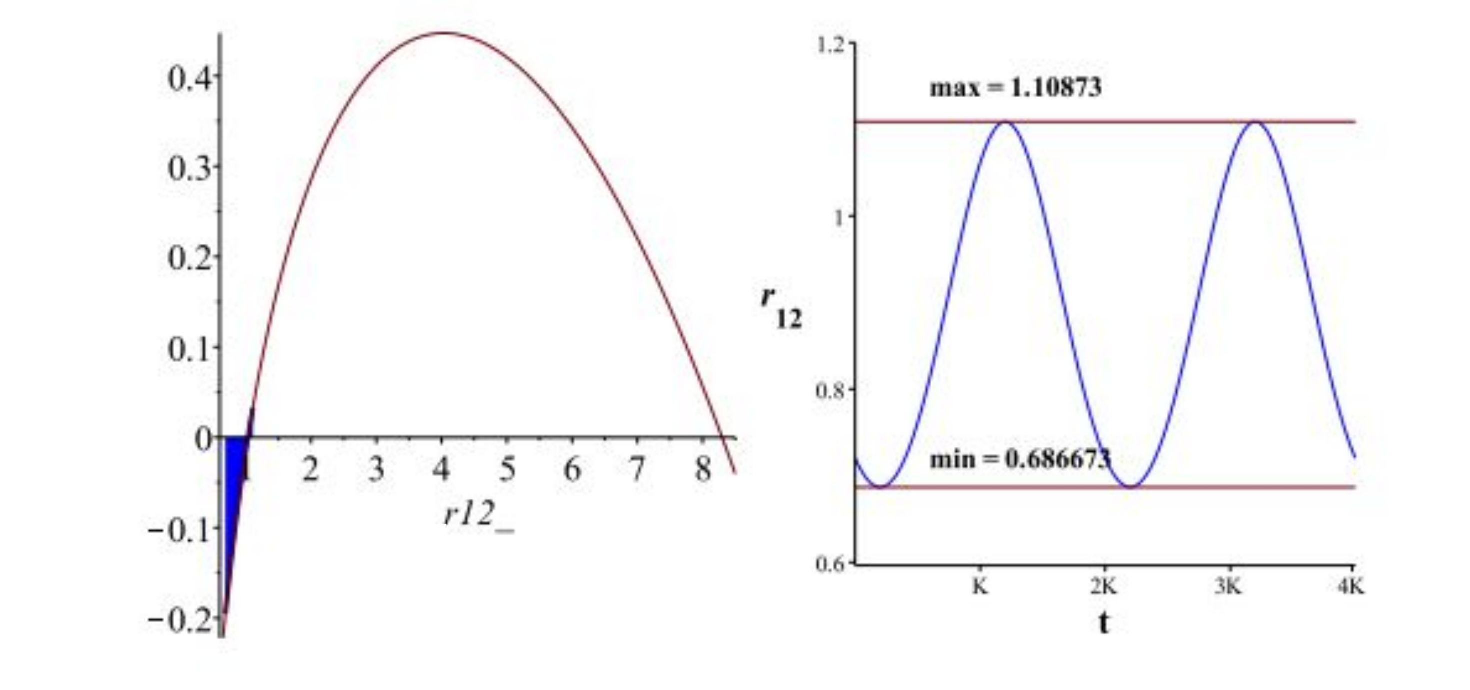
It was found α, β s.t. $\mathcal{V}$ satisfies eight Newton equations for relative motion

$$\frac{d^2}{dt^2} \mathbf{x}_i(t) = -\nabla_{\mathbf{x}_i} \mathcal{V} , \qquad {}_{i=1\ldots4}$$

in seven independent variables r_{ij}

- $\alpha_1 = \frac{1}{4}$, $\beta_1 = 0.04976437360316132382$ (k_1)
- $\alpha_2 = \frac{1}{4}$, $\beta_2 = 0.01536604139547732136$ (k_2)

Pairwise Potential $V(r_{12})$ for k_1



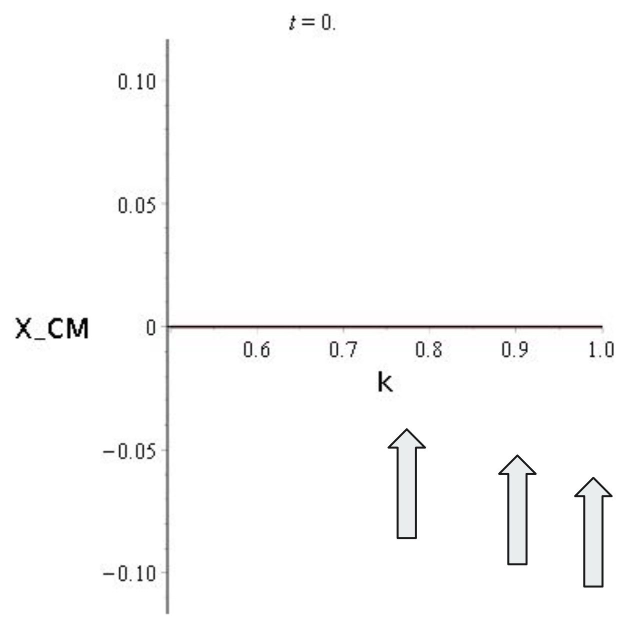
- Five bodies on the plane form degenerate (planar) pentahedron
- Pentahedron is regular (non-degenerate) in 4 (and higher) dimensional space
  It is characterized by 10 edges (relative distances)
- In two-dimensional space seven edges only are independent. There exist three constraints

  ## What they are?

  How to find them?

One constraint is evident: the volume of the pentahedron should be zero. It corresponds to degeneration to 3-dimensional space

7-Bodies on the Lemniscate



7-bodies choreography on the Lemniscate:

$$
\begin{aligned}
x_1(t) &= x(t-3T/7), & y_1(t) &= y(t-3T/7),\\
x_2(t) &= x(t-2T/7), & y_2(t) &= y(t-2T/7),\\
x_3(t) &= x(t-T/7), & y_3(t) &= y(t-T/7),\\
x_4(t) &= x(t), & y_4(t) &= y(t),\\
x_5(t) &= x(t+T/7), & y_5(t) &= y(t+T/7),\\
x_6(t) &= x(t+2T/7), & y_6(t) &= y(t+2T/7),\\
x_7(t) &= x(t+3T/7), & y_7(t) &= y(t+3T/7),
\end{aligned}
$$

$$\mathbf{X}_{\mathrm{CM}}(t) = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_5 + \mathbf{x}_6 + \mathbf{x}_7 = 0 \quad \Rightarrow \quad \text{3 Solutions!}$$

$$k^2 = \begin{cases} 0.5745692809345886540 = k_1^2 \\ 0.8306090006706240710 = k_2^2 \\ 0.9999300005380372772 = k_3^2 \end{cases}$$

Fujiwara *et al.* 2004

7-Bodies on the Lemniscate

7-bodies choreography on the Lemniscate:

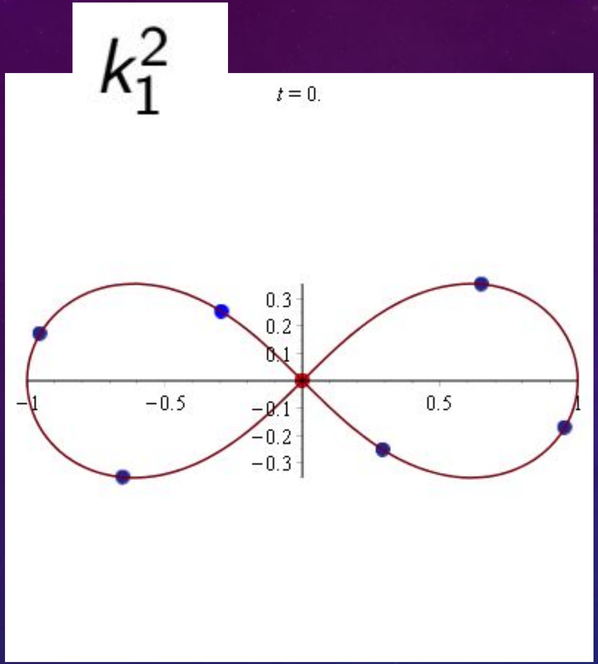
$$\begin{aligned}
x_1(t) &= x(t-3T/7), & y_1(t) &= y(t-3T/7), \\
x_2(t) &= x(t-2T/7), & y_2(t) &= y(t-2T/7), \\
x_3(t) &= x(t-T/7), & y_3(t) &= y(t-T/7), \\
x_4(t) &= x(t), & y_4(t) &= y(t), \\
x_5(t) &= x(t+T/7), & y_5(t) &= y(t+T/7), \\
x_6(t) &= x(t+2T/7), & y_6(t) &= y(t+2T/7), \\
x_7(t) &= x(t+3T/7), & y_7(t) &= y(t+3T/7),
\end{aligned}$$

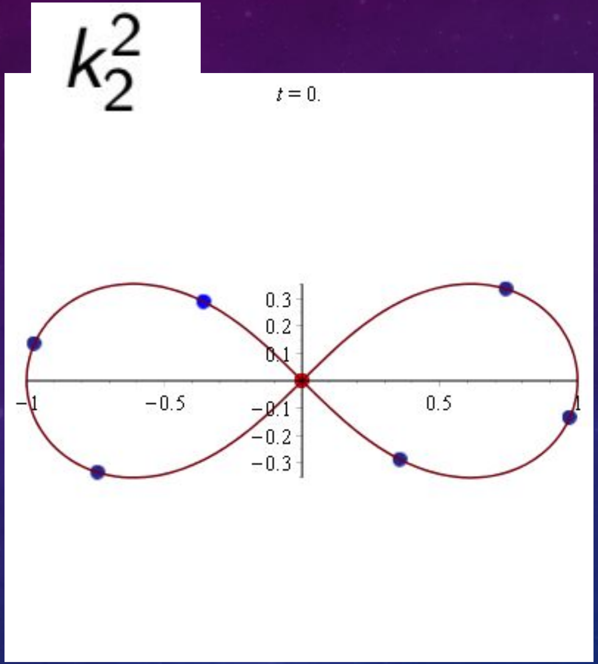
$$\mathbf{X}_{\mathrm{CM}}(t) = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_5 + \mathbf{x}_6 + \mathbf{x}_7 = 0 \quad \Rightarrow \quad \text{3 Solutions!}$$

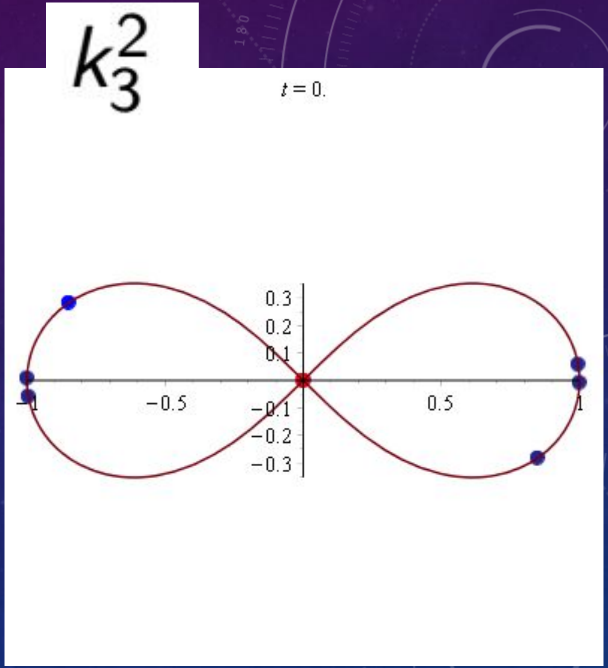
$$k^2 = \begin{cases} 0.5745692809345886 5406 = k_1^2 \\ 0.83060900067062407108 = k_2^2 \\ 0.99993000053803727729 = k_3^2 \end{cases}$$

Fujiwara *et al.* 2004

7-Bodies on the Lemniscate







$$\mathbf{X}_{\mathrm{CM}}(t) = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_5 + \mathbf{x}_6 + \mathbf{x}_7 = 0 \quad \Rightarrow \quad \text{3 Solutions!}$$

$$k^2 = \begin{cases} 0.574569280934588654 06 = k_1^2 \\ 0.830609000670624071 08 = k_2^2 \\ 0.999930000538037277 29 = k_3^2 \end{cases}$$

7 Bodies on the Lemniscate. Conserved quantities

Conserved quantities:

1. $\sum \mathbf{x}_i \times \mathbf{v}_i = 0$ Angular Momentum
2. $E = \mathcal{T} + \mathcal{V}$ Total Energy (dependable)
$$= \begin{cases} 0.16423442473755265532 \text{ for } k_1 \\ 0.78148931963620224401 \text{ for } k_2 \\ 0.50215171876269155046 \text{ for } k_3 \end{cases}$$
3. $$I_{HR}^{(7)} = 7\sum \mathbf{x}_i^2 = \sum_{i<j} r_{ij}^2 = \begin{cases} 22.88340861420164459 0\text{ for } k_1 \\ 25.66279892437685127 2\text{ for } k_2 \\ 39.10341565088814854 0\text{ for } k_3 \end{cases}$$
Hyper-radius (Moment of Inertia)
4. $$\mathcal{T} = \frac{1}{2}\sum \mathbf{v}_i^2 = \begin{cases} 1.6281143338790436808 \text{ for } k_1 \\ 1.1439748352286144920 \text{ for } k_2 \\ 0.3536449566151709007 7\text{ for } k_3 \end{cases}$$
Kinetic Energy

7 Bodies on the Lemniscate. Conserved quantities

Conserved quantities:

5. $$\tilde{\mathfrak{T}} = \mathbf{v}_1^2\mathbf{v}_2^2\mathbf{v}_3^2\mathbf{v}_4^2\mathbf{v}_5^2\mathbf{v}_6^2\mathbf{v}_7^2 = \begin{cases} 0.00466005751814023926 \ (k_1) \\ 0.00024297642722229551 \ (k_2) \\ 1.18891671546594582 5 \times 10^{-15} \ (k_3) \end{cases}$$

7-Bodies. Particular constants of motion

It was found

- 7-Body

$$I_1^{(7)} = \begin{cases} r_{12}^2 r_{23}^2 r_{34}^2 r_{45}^2 r_{56}^2 r_{67}^2 r_{17}^2 = 0.26362178303408707110 \ (k_1) \\ r_{13}^2 r_{35}^2 r_{57}^2 r_{27}^2 r_{24}^2 r_{46}^2 r_{16}^2 = 2.6489374483809056078 \ (k_2) \\ r_{14}^2 r_{47}^2 r_{37}^2 r_{36}^2 r_{26}^2 r_{25}^2 r_{15}^2 = 482.72504286636644935 \ (k_3) \end{cases}$$

and

$$I_2^{(7)} = \begin{cases} r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{45}^2 + r_{56}^2 + r_{67}^2 + r_{17}^2 = 3.3066909304564091707 \ (k_1) \\ r_{13}^2 + r_{35}^2 + r_{57}^2 + r_{27}^2 + r_{24}^2 + r_{46}^2 + r_{16}^2 = 9.5236213317249233435 \ (k_2) \\ r_{14}^2 + r_{47}^2 + r_{37}^2 + r_{36}^2 + r_{26}^2 + r_{25}^2 + r_{15}^2 = 20.460348437174532420 \ (k_3) \end{cases}$$

- $J_i(k) = \mathbf{v}_i^2 + (k^2 - 1/2)\mathbf{x}_i^2 = 1/2, \quad i = 1, 2, \dots 7$

$\sum^7 J_i(k_0) = 2\,\mathcal{T} + \frac{1}{7}(k_{1,2}^2 - \frac{1}{2}) I_{HR}^{(7)}$

In total 7 constants of motion

7 Bodies on the Lemniscate. Potential

It was found

$$V = \alpha \log I_1^{(7)} - \beta I_{\mathrm{HR}}^{(7)}$$

$$V = \begin{cases} \alpha_1 \left\{ \log r_{12}^2 + \log r_{23}^2 + \log r_{34}^2 + \log r_{45}^2 + \log r_{56}^2 + \log r_{67}^2 + \log r_{17}^2 \right\} \\ \alpha_2 \left\{ \log r_{13}^2 + \log r_{35}^2 + \log r_{57}^2 + \log r_{27}^2 + \log r_{24}^2 + \log r_{46}^2 + \log r_{16}^2 \right\} \\ \alpha_3 \left\{ \log r_{14}^2 + \log r_{47}^2 + \log r_{37}^2 + \log r_{36}^2 + \log r_{26}^2 + \log r_{25}^2 + \log r_{15}^2 \right\} \end{cases} - \beta_{1,2,3} \sum_{i<j} r_{ij}^2 ,$$

Potential:

It was found α, β s.t. V satisfies twelve Newton equations of relative motion

$$\frac{d^2}{dt^2} \mathbf{x}_i(t) = -\nabla_{\mathbf{x}_i} V \,, \qquad i{=}1\ldots6$$

in 11 independent variables r_{ij}

- $\alpha_1 = \frac{1}{4}$, $\beta_1 = 0.005326377209613475282\!6\ (k_1)$
- $\alpha_2 = \frac{1}{4}$, $\beta_2 = 0.023614928619330290764\ (k_2)$
- $\alpha_3 = \frac{1}{4}$, $\beta_3 = 0.035709285752716948625\ (k_3)$

- Is the Lemniscate for 7-body (particularly maximally) superintegrable trajectory?
  (Does T-conjecture (2013) hold?)

  For 7-body case maximally superintegrable trajectory is characterized by 23 constants of motion,
  we found 12 so far,

- What are the missing particular constants of motion?

General case: $(2n+1)$ bodies on the lemniscate

Dimension of relative motion space is $4n$

$$
\begin{aligned}
x_1(\tau) &= x(\tau - nT/(2n+1)) & y_1(\tau) &= y(\tau - nT/(2n+1)) \\
x_2(\tau) &= x(\tau - (n-1)T/(2n+1)) & y_2(\tau) &= y(\tau - (n-1)T/(2n+1)) \\
&\dots & &\dots \\
x_n(\tau) &= x(\tau) & y_n(\tau) &= y(\tau) \\
&\dots & &\dots \\
x_{2n}(\tau) &= x(\tau + (n-1)T/(2n+1)) & y_{2n}(\tau) &= y(\tau + (n-1)T/(2n+1)) \\
x_{2n+1}(\tau) &= x(\tau + nT/(2n+1)) & y_{2n+1}(\tau) &= y(\tau + nT/(2n+1))
\end{aligned}
$$

$$
x_j = x\left(\tau - (n+1-j)\frac{T}{2n+1}\right) \qquad y_j = y\left(\tau - (n+1-j)\frac{T}{2n+1}\right), \quad j = 1 \dots 2n+1
$$

$$
\mathbf{x}_j = (x_j, y_j)
$$

$$
\mathbf{X}_{\mathrm{CM}}(t) = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_{2n+1} = 0\,,
$$

- Conjecture: $\exists$ n solutions for k: $k_j(n)\,, \quad j = 1 \dots n$
- Conjecture: $\lim_{n\to\infty} k_j^2(n) \to 1$

$(2n+1)$-bodies on the Lemniscate. Conserved quantities

Conserved quantities:

1. $\sum_{i=1}^{(2n+1)} \mathbf{x}_i \times \mathbf{v}_i = 0$ Angular Momentum
2. $E = \mathcal{T} + \mathcal{V}$ Total Energy (dependable)
3. $I_{HR}^{(2n+1)} = (2n+1)\sum_{i=1}^{(2n+1)} \mathbf{x}_i^2 = \sum_{i<j} r_{ij}^2$

   Hyper-radius (Moment of Inertia) - conjecture

4. $\mathcal{T} = \frac{1}{2}\sum_{i=1}^{(2n+1)} \mathbf{v}_i^2$ Kinetic Energy - conjecture
5. $\tilde{\mathcal{T}} = \mathbf{v}_1^2 \mathbf{v}_2^2 \ldots \mathbf{v}_{2n+1}^2$ - conjecture
6. $J_i(k) = \mathbf{v}_i^2 + (k^2 - 1/2)\mathbf{x}_i^2 = 1/2, \quad i = 1, 2, \ldots (2n+1)$

$$\sum_{i=1}^{(2n+1)} J_i(k_0) = 2\,\mathcal{T} + \frac{1}{2n+1}\left(k_{1\ldots n}^2 - \frac{1}{2}\right) I_{HR}^{(2n+1)}$$

$(2n+1)$-bodies. Particular constants of motion

Conjecture

- $(2n+1)$-body

$$I_1^{(2n+1)} = \begin{cases} r_{12}^2 r_{23}^2 r_{34}^2 \dots r_{2n,2n+1}^2 r_{1,(2n+1)}^2 \text{ for } (k_1) \\ \dots \text{ for } (k_n) \end{cases}$$

(product of $(2n+1)$ relative distances squared), and

$$I_2^{(2n+1)} = \begin{cases} r_{12}^2 + r_{23}^2 + r_{34}^2 \dots + r_{2n,2n+1}^2 + r_{1,(2n+1)}^2 \text{ for } (k_1) \\ \dots (k_n) \end{cases}$$

(sum of $(2n+1)$ relative distances squared)

$(2n+1)$-bodies on the Lemniscate: Potential

Conjecture

$$V = \alpha \log I_1^{(2n+1)} - \beta I_{\mathrm{HR}}^{(2n+1)}$$

$$V = \begin{cases} \alpha_1 \{\log r_{12}^2 + \log r_{23}^2 \ldots + \log r_{2n,2n+1}^2 + \log r_{1,2n+1}^2\} \\ \ldots \end{cases} - \beta_{1,2,\ldots 2n+1} \sum_{i<j} r_{ij}^2 ,$$

Potential (conjecture):

There exist α, β s.t. V satisfies $4n$ Newton equations of relative motion

$$\frac{d^2}{dt^2}\mathbf{x}_i(t) = -\nabla_{\mathbf{x}_i} V \,, \qquad {}_{i=1\ldots 2n-1}$$

in $4n-1$ independent variables r_{ij}

- $\alpha_1 = \frac{1}{4}$, $\beta_1 = \ldots$ (k_1)
- $\alpha_n = \frac{1}{4}$, $\beta_n = \ldots$ (k_n)

  For $n \to \infty$, $\beta \to 0$

Conclusion

- choreographies of 3,5,7, and possibly any odd number of bodies on the **same** Lemniscate exist.
- For 5 bodies there exist 2 possible choreographies corresponding to two possible configurations of 5/10 relative distances out of 7 independent (two solutions for k such that $\mathbf{X}_{\mathrm{CM}}(t) = 0$).
- For 7 bodies there exist 3 possible choreographies corresponding to three possible configurations of 7/21 relative distances out of 11 independent (three solutions for k such that $\mathbf{X}_{\mathrm{CM}}(t) = 0$).
- Potentials $V_{3,5,7}$ were found such that Newton equations are satisfied, *i.e.* True choreographies!

Conclusion

- CONJECTURE: for $(2n+1)$-body case at $n \to \infty$ one of the potentials

$$V_{(2n+1)} = \frac{1}{4} \log \left(\prod_{i=1}^{2n+1} r_{i,i+1}^2 \right)$$

$$k^2 \to 1$$

Planar two-dimensional logarithmic Newtonian gas?

- **In n-body Newton problem are choreographies the (super)-integrable trajectories?**

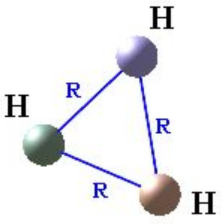
Quarkonium

$$V = \frac{a}{r} + b\,r\,, \qquad b > 0$$

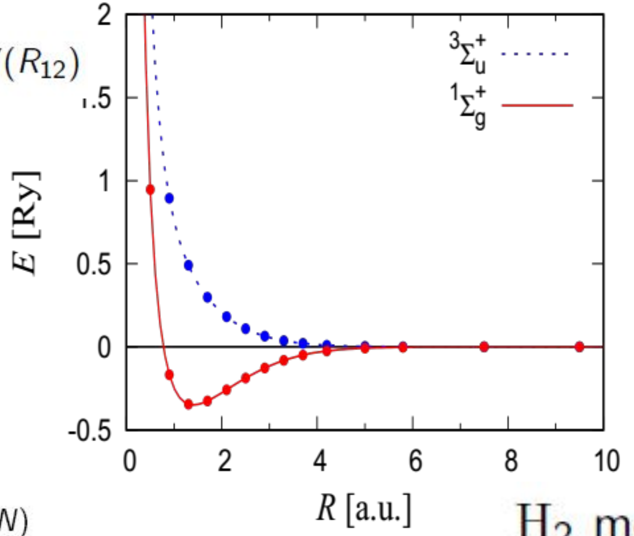
Do exist Figure-Eight trajectory?

math proves the existence (!)

H_3 Molecule



$V(R_{12})$



H_2 molecule

$$V = V(R_{12}) + V(R_{13}) + V(R_{23}) + V_3(R_{12}, R_{13}, R_{23})$$

$$V(R) = \frac{1}{R} P(N|N+3) + e^{-R} Q(N+1|N)$$

Does Figure-Eight trajectory exist?