Equivalence Between Formulations in Cosmological Perturbation Theory: The primordial magnetic fields as an example.
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Abstract

Nowadays, Cosmological Perturbation Theory is a standard and useful tool in theoretical cosmology [1]. In this work, we compare the 1+3 covariant formalism in perturbation theory (Ellis) [2] to the gauge invariant approach (Bruni et al.[3]), and we show the equivalence of these formalisms to fix the choice of the perturbed variables (gauge choice) in magnetogenesis. We analyze the evolution of primordial magnetic fields through perturbation theory and we discuss the similarities and differences between these two approaches. We get the Maxwell’s equations and show a cosmic dynamo like equation written in Poisson gauge. The comparison of the evolution of primordial magnetic fields. Finally, prospects around these formalisms in the study of magnetogenesis are discussed.

The gauge problem in perturbation theory

Cosmological perturbation theory help us to find approximated solutions of the Einstein field equations through small deviations from an exact solution [3]. The gauge invariant formalism is developed into two space-times, one is the real space-time ($M, g_{\mu\nu}$) which describes the perturbed universe and the other one is the background space-time ($M_0, g^0_{\mu\nu}$) which is an idealization and is taken as reference to generate the real space-time. A mapping between these two space-times called gauge choice given by a function $\chi: M_0(p) \rightarrow M(p)$ for any point $p \in M_0$ and $p \in M$, which generates a pull-back $\chi^* : \mathcal{M} \rightarrow \mathcal{M}_0$ thus, points on the real and background space-time can be compared through $\chi$.

Figure 1: Gauge transformation.

General covariant states that there is no preferred coordinate system in nature and it introduce a gauge in perturbation theory. This gauge is an unphysical degree of freedom and we have to fix the gauge or to extract some invariant quantities to have physical results. Then, the perturbation for $\chi$ is defined as

$$g_{\mu\nu} - \gamma_{\mu\nu} = L_{\chi} \gamma_{\mu\nu}$$

We see that the perturbation $\Delta \gamma$ is completely dependent of the gauge choice because the mapping determines the representation on $M_0(1)$. However, one can also choose another correspondence $\chi$ between these space-times so that $\chi : M_0(q) \rightarrow M[q]$, $p \neq q$. The freedom to choose different correspondences generates an arbitrariness in the value of $\Delta \gamma$ at any space-time point $p$, which is called gauge problem.

Given a tensor field $\gamma$, the relations between first and second order perturbations of $\gamma$ in two different gauges are

$$\Delta \gamma^{\alpha\beta} = \gamma^{\alpha\beta} - \gamma^{\alpha\beta} = L_{\chi} \gamma^{\alpha\beta}$$

A tensor field $\gamma$ is gauge-invariant to order $n \geq 1$ if $L_{\chi} \gamma^{\alpha\beta} = 0$, for any vector field and $n < n$. This vector field can be split in their time and space part

$$\xi^{(1)} = w^{(1)} + \frac{1}{a} \gamma^{(1)}$$

Here $\xi^{(1)}$ and $\nu^{(1)}$ are arbitrary scalar functions, and we have $\partial \xi^{(1)} = \partial _{\chi}$.

The function $\alpha_{ij}$ determines the choice of time constant hyper-surfaces, while $\alpha^{(1)}_{ij}$ and $\nu^{(1)}$ fix the spatial coordinates within these hypersurfaces.

Gauge invariant variables at first order

We consider the perturbations about a FLRW background, so the metric tensor is given by $\gamma = \mu + \sum_{i=1}^{\infty} \frac{\mu}{r^{2i}}$.

$$g_{\mu \nu} = \delta_{\mu \nu} + \sum_{i=1}^{\infty} \frac{\mu}{r^{2i}}$$

The electric field with the current should be zero, thus the evolution of primordial magnetic fields. In the case of a homogeneous collapse, $B = \gamma^{\alpha} \gamma_{\alpha}$ there is an amplification of the magnetic field in places where gravitational collapse take place. In eq.(13), the energy density magnetic field at second order transforms as $\mathcal{E}_{\gamma}^{(2)} = B_{\chi}^{(2)} - \alpha_{\chi} \mathcal{E}_{\gamma}^{(0)} + 2 B_{\chi}^{(1)} \mathcal{E}_{\gamma} + B_{\chi}^{(0)} \mathcal{E}_{\chi}^{(1)} + 2 B_{\chi}^{(2)} \mathcal{E}_{\chi}^{(0)}$. Finally, we relate quantities in the 1+3 covariant formalism and in the invariant approach showed above. In the covariant formalism quantities are projected down onto spatial $h_{\mu\nu}$, relative to the 4-velocity of the fluid. This suggests that the quantities constructed in this way are closely related to quantities gauge invariant using the comoving gauge [4]. The comoving magnetic density gradient is defined as $B_{\chi}^{(2)} = \frac{1}{a} \frac{d}{dt} \nabla B_{\mu}$, with $\mu_{\chi}^{(2)} \equiv \mu_{\chi} + u_{\chi} u_{\chi}$.

Now, we substitute the 4-velocity at first order found in gauge invariant approach $u_{\chi} = - (1 + \psi, \partial_\mu B_\mu)$, we obtain the following relation

$$u_{\chi} \equiv \frac{1}{a} \frac{d}{dt} \nabla B_{\mu}$$

where the comoving gauge is used (which introduces a family of world lines orthogonal to the 3-D spatial sections) given by $a \rightarrow \partial_\mu B_\mu$, in eq.(10), it is derived a similar to expression as it was found in 1+3 covariant formalism, it implies an equivalence in both formalisms. Now, if we study this equivalence at second order we must impose $u_{\chi}^{(2)} = 0$ to provide a covariant description. In this case the 4-velocity at second order is

$$u_{\chi}^{(2)} = \frac{1}{a} \frac{d}{dt} \nabla B_{\mu}$$

in [5] is shown vector field that determines the gauge comoving at second order. In this case one must take into account that 4-velocity must be zero and choose appropriately the 3D spatial section through of $\beta^{(2)}$ and $d^{(2)}$.

References