Equivalence Between Formulations in Cosmological Perturbation Theory: The primordial magnetic fields as an example.

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Abstract

Nowadays, Cosmological Perturbation Theory is a standard and useful tool in theoretical cosmology [1]. In this work, we compare the 1+3 covariant formalism in perturbation theory (Ellis) [2] to the gauge invariant approach (Bruni et. al.)[3], and we show the equivalence of these formalisms to fix the choice of the perturbed variables (gauge choice) in magnetogenesis. We analyze the evolution of primordial magnetic fields through perturbation theory and we discuss the similarities and differences between these two approaches. We get the Maxwell's equations and show a cosmic dynamo like equation written in Poisson gauge, computing the evolution of primordial magnetic fields. Finally, prospects around these formalisms in the study of magnetogenesis are discussed.

The gauge problem in perturbation theory

Cosmological perturbation theory help us to find approximated solutions of the Einstein field equations through small desviations from an exact solution [3]. The gauge invariant formalism is developed into two space-times, one is the real space-time $(\mathcal{M}, g_{\alpha\beta})$ which describes the perturbed universe and the other one is the background space-time $(\mathcal{M}_0, g^{(0)}_{\alpha\beta})$ which is an idealization and is taken as reference to generate the real space-time. A mapping between these space-times called *gauge choice* given by a function $\mathcal{X}: \mathcal{M}_0(p) \longrightarrow \mathcal{M}(\bar{p})$ for any point $p \in \mathcal{M}_0$ and $\bar{p} \in \mathcal{M}$, which generates a pull-back $\boxed{\mathcal{X}^*: \begin{array}{c}\mathcal{M} \\ T^*(\bar{p})\end{array} \longrightarrow \begin{array}{c}\mathcal{M}_0 \\ T^*(p)\end{array}}$, thus, points on the real and background space-time can be compared through of $\mathcal{X}$.

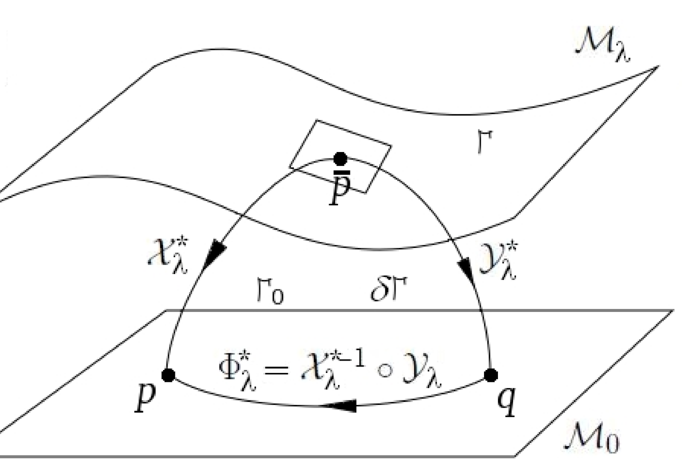

Figure 1: Gauge transformation.

General covariance states that there is no preferred coordinate system in nature and it introduce a gauge in perturbation theory. This gauge is an unphysical degree of freedom and we have to fix the gauge or to extract some invariant quantities to have physical results. Then, the perturbation for Γ is defined as

$$\Gamma(\bar{p}) = \Gamma_0(p) + \delta\Gamma(p)\,. \tag{1}$$

We see that the perturbation $\delta\Gamma$ is completely dependent of the gauge choice because the mapping determines the representation on $\mathcal{M}_0$ of $\Gamma(\bar{p})$. However, one can also choice another correspondence $\mathcal{Y}$ between these space-times so that $\mathcal{Y}: \mathcal{M}_0(q) \to \mathcal{M}(\bar{p})$, $(p \neq q)$. The freedom to choose different correspondences generate an arbitrariness in the value of $\delta\Gamma$ at any space-time point p, which is called *gauge problem*.

Given a tensor field Γ, the relations between first and second order perturbations of Γ in two different gauges are

$$\delta\Gamma^{\mathfrak{X}} - \delta\Gamma^{\mathfrak{Y}} = L_{\xi_1}\Gamma_0, \tag{2}$$

$$\delta^2\Gamma^{\mathfrak{Y}} - \delta^2\Gamma^{\mathfrak{X}} = 2L_{\xi_1}\left(\delta\Gamma^{\mathfrak{X}}\right) + \left(L^2_{\xi_1} + L_{\xi_2}\right)\Gamma_0. \tag{3}$$

A tensor field Γ is gauge-invariant to order $n \geq 1$ if $L_\xi \delta^k \Gamma = 0$, for any vector field and $\forall k < n$. This vector field can be splited in their time and space part

$$\xi^{(r)}_\mu \to \left(\alpha^{(r)}, \partial_i\beta^{(r)} + d^{(r)}_i\right), \tag{4}$$

here $\alpha^{(r)}$ and $\beta^{(r)}$ are arbitrary scalar functions, and we have $\partial^i d^{(r)}_i = 0$.

The function $\alpha_{(r)}$ determines the choice of time constant hypersurfaces, while $\partial_i\beta^{(r)}$ and $d^{(r)}_i$ fix the spatial coordinates within these hypersurfaces.

Gauge invariant variables at first order

We consider the perturbations about a FLRW background, so the metric tensor is given by :

$$\mu = \mu_0 + \sum_{r=1}^{\infty}\frac{1}{r!}\delta^r\mu, \quad u^\alpha = \frac{1}{a}\left(\delta^\alpha_0 + \sum_{r=1}^{\infty}\frac{1}{r!}v^\alpha_{(r)}\right) \tag{5}$$

$$g_{00} = -a^2(\tau)\left(1 + 2\sum_{r=1}^{\infty}\frac{\psi^{(r)}}{r!}\right), \quad g_{oi} = a^2(\tau)\sum_{r=1}^{\infty}\frac{\omega^{(r)}_i}{r!}$$
$$g_{ij} = a^2(\tau)\left[\left[\left(1 - 2\sum_{r=1}^{\infty}\frac{1}{r!}\phi^{(r)}\right)\right]\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}\chi^{(r)}_{ij}\right] \tag{6}$$

It is found the scalar and vector *gauge invariant variables* at first order given by

$$\Psi^{(1)} \equiv \psi^{(1)} + \frac{\left(\mathcal{S}^{||}_{(1)}a\right)'}{a}, \quad \Phi^{(1)} \equiv \phi^{(1)} + \frac{\nabla^2\chi^{(1)}}{6} - H\mathcal{S}^{||}_{(1)},$$
$$\Delta^{(1)} \equiv \mu_{(1)} + \left(\mu_{(0)}\right)'\mathcal{S}^{||}_{(1)}, \quad \Delta^{(1)}_P \equiv P_{(1)} + \left(P_{(0)}\right)'\mathcal{S}^{||}_{(1)},$$
$$\mathcal{V}^i_{(1)} \equiv \omega^i_{(1)} + v^i_{(1)}, \tag{7}$$

with $\mathcal{S}^{||}_{(1)} \equiv \left(\omega^{||(1)} - \frac{\left(\chi^{||(1)}\right)'}{2}\right)$ the scalar contribution of the shear (associated with $\alpha^{(1)}$). Using the Einstein's equation at first order, it is expressed the evolution of geometrical variables ϕ and ψ, and the conservation's equations entails the evolution of energy density $\Delta^{(1)}$

$$\begin{aligned}&\left(\Delta^{(1)}\right)' + 3H\left(\Delta^{(1)}_P + \Delta^{(1)}\right) - 3\left(\Phi^{(1)}\right)'\left(P_{(0)} + \mu_{(0)}\right)\\ &+ \left(P_{(0)} + \mu_{(0)}\right)\nabla^2 v^{(1)} - 3H\left(P_{(0)} + \mu_{(0)}\right)'\mathcal{S}^{||}_{(1)}\\ &- \left(\left(\mu_{(0)}\right)'\mathcal{S}^{||}_{(1)}\right)' + \left(P_{(0)} + \mu_{(0)}\right)\left(-\frac{1}{2}\nabla^2\chi^{(1)} + 3H\mathcal{S}^{||}_{(1)}\right)'\\ &- \left(P_{(0)} + \mu_{(0)}\right)\nabla^2\left(\frac{1}{2}\chi^{||(1)}\right)' = 0,\end{aligned} \tag{8}$$

and peculiar velocity $\mathcal{V}_{(1)}$ given by

$$\begin{aligned}&\left(\mathcal{V}^{(1)}_i\right)' + \frac{\left(\mu_{(0)} + P_{(0)}\right)'}{\left(\mu_{(0)} + P_{(0)}\right)}\mathcal{V}^{(1)}_i - 4H\mathcal{V}^{(1)}_i + \partial_i\Psi^{(1)}\\ &- \frac{\partial_i\left(\Delta^{(1)}_P - \left(P_{(0)}\right)'\mathcal{S}^{||}_{(1)}\right) + \partial_l\Pi^{(1)l}_{(fl)i}}{\left(\mu_{(0)} + P_{(0)}\right)} - \partial_i\frac{1}{a}\left(\mathcal{S}^{||}_{(1)}a\right)' = 0.\end{aligned} \tag{9}$$

Weakly magnetized FLRW-background

We allow the presence of a weak magnetic field into our FLRW space-time with the property $B^2_{(0)} \ll \mu_{(0)}$ and must be sufficiently random to satisface $\langle B_i \rangle = 0$ and $\left\langle B^2_{(0)} \right\rangle \neq 0$ to ensure that symmetries and the evolution of the background remain unaffected (we assume that at zero order the magnetic field has been generated by some random process which is statistically homogeneous so that $B^2_{(0)}$ just time depending and $\langle .. \rangle$ denotes the expectation value) [4]. We work under MHD approximation, thus, in large scales the plasma is globally neutral, charge density is neglected and the electric field with the current should be zero, thus the only zero order magnetic variable is $B^2_{(0)}$. At first order it is obtained a gauge invariant term which describes the magnetic energy density

$$\Delta^{(1)}_{mag} \equiv B^2_{(1)} + \left(B^2_{(0)}\right)'\alpha^{(1)}. \tag{10}$$

Another gauge invariant variables are the 3-current J, the charge density ϱ and the electric and magnetic fields, because they vanish in the background.

At first order, the electric field and the current become nonzero and assuming the ohmic current is not neglected, we find the Ohm's law

$$J^{(1)}_i = \sigma\left[E^{(1)}_i + \left(\mathcal{V}^{(1)} \times B^{(0)}\right)_i\right]. \tag{11}$$

The perturbed equations for the metric and electromagnetic fields are given by

$$\partial_i E^i_{(1)} = a\varrho_{(1)}, \qquad \epsilon^{ilk}\partial_l B^{(1)}_k = \left(a^2 E^i_{(1)}\right)' + a^3 J^i_{(1)},$$
$$\partial^i B^{(1)}_i = 0, \qquad \left(a^2 B^{(1)}_k\right)' + a^2\epsilon^{ij}_{\ \ k}\partial_i E^{(1)}_j = 0. \tag{12}$$

Now using eq.(12) together with the Ohm's law, we get a cosmic dynamo like equation which describes the evolution of density magnetic field at first order in the Poisson gauge

$$\begin{aligned}&\frac{d\Delta^{(2)}_{(mag)}}{dt} + 4H\Delta^{(2)}_{(mag)} + \frac{2}{3}\Delta^{(1)}_{(mag)}\partial_l\mathcal{V}^l_{(1)} + 2\eta\left[-B^{(1)}\cdot\nabla^2 B^{(1)}\right.\\ &-B^{(1)}_k\cdot\left(\nabla\times\left(\frac{dE_{(1)}}{dt} + 2HE_{(1)}\right)\right)^k - \frac{\Delta^{(1)}_{(mag)}}{2}\nabla^2\left(\Psi^{(1)} - 3\Phi^{(1)}\right)\\ &\left.+B^k_{(1)}\left(B^{(0)}\cdot\nabla\right)\partial_k\left(\Psi^{(1)} - 3\Phi^{(1)}\right)\right] = -2\Pi^{(1)}_{ij(em)}\sigma^{ij}_{(1)},\end{aligned} \tag{13}$$

where we use the Lagrangian coordinates which are comoving with the local Hubble flow and magnetic field lines are frozen into the fluid ($\frac{d}{dt} = \frac{\partial}{\partial t} + \mathcal{V}^i_{(1)}\partial_i$), σ and Π are the shear and stress Maxwell tensor respectively. Thus, the perturbations in the space-time play an important role in the evolution of primordial magnetic fields. In the case of a homogeneous collapse, $B \sim \mathcal{V}^{-\frac{2}{3}}$ there is an amplification of the magnetic field in places where gravitational collapse take place. In eq.(13), the energy density magnetic field at second order transforms as

$$\begin{aligned}\Delta^{(2)}_{(mag)} &= B^2_{(2)} + \alpha_{(1)}\left(B^{2\prime\prime}_{(0)}\alpha_{(1)} + B^{2\prime}_{(0)}\alpha'_{(1)} + 2B^{2\prime}_{(1)}\right)\\ &+\xi^i_{(1)}\left(B^{2\prime}_{(0)}\partial_i\alpha^{(1)} + 2\partial_i B^2_{(1)}\right) + B^{2\prime}_{(0)}\alpha_{(2)}.\end{aligned} \tag{14}$$

Finally, we relate quantities in the 1+3 covariant formalism and in the invariant approach showed above. In the covariant formalism quantities are projected down onto spatial $h_{\alpha\beta}$, relative to the 4-velocity of the fluid. This suggests that the quantities constructed in this way are closely related to quantities gauge invariant using the comoving gauge [4]. The comoving magnetic density gradient is defined as

$$\mathcal{B}_\mu \equiv \frac{a}{B^2}h^\lambda_\mu\nabla_\lambda B^2, \quad \text{with} \quad h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu\,. \tag{15}$$

Now, we substitute the 4-velocity at first order found in gauge invariant approach $u_\mu = a\left(-(1+\psi),\ \partial_i(v^{\|} + \omega^{\|})\right)$, we obtain the following relation

$$h^\lambda_\mu\nabla_\lambda B^2 = \partial_i\left(B^2_{(1)} + B^{2\prime}_{(0)}\left(v^{\|} + \omega^{\|}\right)\right), \tag{16}$$

if the comoving gauge is used (which introduces a family of world lines orthogonal to the 3-D spatial sections) given by $\alpha \to v^{\|} + \omega^{\|}$ in eq. (10), it is derived a similar to expression as it was found in 1+3 covariant formalism, it implies an equivalence in both formalisms. Now, if we study this equivalence at second order, we must impose $u^{(2)}_i = 0$ to provide a covariant description. In this case the 4-velocity at second order is

$$\frac{u^{(2)}_i}{a} = \left[\frac{\left(v^{(2)}_i + \omega^{(2)}_i\right)}{2} - 2v^{(1)}_i\phi^{(1)} - \omega^{(1)}_i\psi_{(1)} + v^j_{(1)}\chi^{(1)}_{ij}\right] \tag{17}$$

in [5] is shown vector field that determines the gauge comoving at second order. In this case one must take into account that 4-velocity must be zero and choose appropriately the 3D spatial section through of $\beta^{(2)}$ and $d^{(2)}_i$.

References

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