MagnetoHydroDynamic equilibria in barotropic stars

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Baro... what?

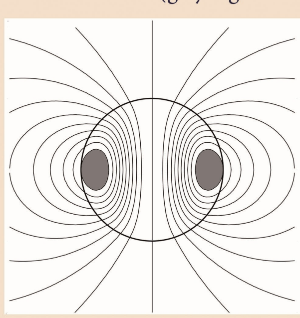
Barotropic equations of state, where pressure is a function solely of density, are usually assumed to describe the matter within magnetic stars in ideal magnetohydrodynamical (MHD) equilibrium. (Yoshida & Eriguchi 2006, Haskell et al. 2008, Lander & Jones 2009). Barotropy strongly restricts the range of possible equilibrium configurations, and strictly does not represent the realistic stably stratified matter within these objects. Stable stratification is likely to be an essential ingredient to suppress the magnetic instabilities (Reisenegger 2009). Thus, it is interesting to verify if barotropic equilibria are stable or not, so that they can have a chance to be used in modelling the known long-lived magnetic fields present in some stellar objects. These equilibria, involving both poloidal and toroidal magnetic field components, are described by the so-called Grad-Shafranov equation, a nonlinear, partial differential equation with two free functions (Grad & Rubin 1958, Shafranov 1966). Here, we present a new finite-difference code developed in order to solve this equation for arbitrary choices of these functions.

The model

In the ideal MHD approximation, a magnetic star may be considered as a perfectly conducting spherical fluid in dynamical equilibrium described by the Euler equation. If axial symmetry is assumed, all quantities are independent of the azimuthal coordinate, and the magnetic field may be expressed as the sum of a *poloidal* (meridional field lines crossing the *pol*es of the star) component and a *toroidal* (azimuthal field lines in a *tor*us within the star) component, each determined by a single scalar function,

$$\mathbf{B} \equiv \mathbf{B}_{\rm pol} + \mathbf{B}_{\rm tor} = \boldsymbol{\nabla}\alpha(r,\theta) \times \boldsymbol{\nabla}\phi + \beta(r,\theta)\boldsymbol{\nabla}\phi,$$

which turn out to be constant along their respective field lines (Chandrasekhar & Prendergast 1956). Under this symmetry, the azimuthal component of the Lorentz force per unit volume must vanish, which implies a functional relation between these functions, $\beta(r,\theta) = \beta(\alpha(r,\theta))$. In this way, both α and β are constant along field lines, and the toroidal field may only lie in regions where the poloidal field lines close within the star (gray regions in the figure below right).



A barotropic equation of state, $P = P(\rho)$, implies that the Lorentz force per unit mass must be the gradient of some arbitrary scalar function $\chi(r,\theta)$, which turns out to be function of α as well, $\chi(r,\theta) = \chi(\alpha(r,\theta))$ as well. Replacing this formalism in the equilibrium equation, a nonlinear PDE is obtained, the so-called Grad-Shafranov equation, depending on two arbitrary magnetic functions: $\beta(\alpha)$ *and* $\chi(\alpha)$:

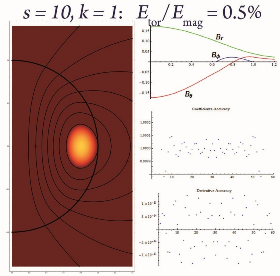
Grad-Shafranov equation (Grad & Rubin 1958, Shafranov 1966)

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\alpha}{\partial\theta}\right) + \beta(\alpha)\beta'(\alpha) + 4\pi\rho r^2\sin^2\theta\,\chi'(\alpha) = 0,$$

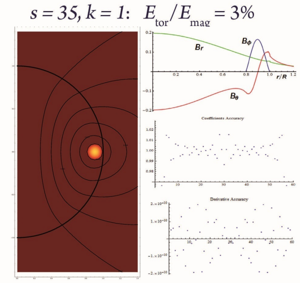
Numerical results

We have implemented a finite-difference code to solve the G-S equation for arbitrary choices of the functions $\beta(\alpha)$ *and* $\chi(\alpha)$. In this case we have chosen the form $\beta(\alpha) = s(\alpha - \alpha_s)^\lambda$ *for* $\alpha > \alpha_s$, and *zero* otherwise, where α_s is the value of α at the larger poloidal field line closing within the star, i.e., at the equator, recalling that the toroidal field must be confined to lie inside this field line. Outside the star, α corresponds to a multipolar expansion which is solution of the G-S equation with $\beta = 0$ and $\rho = 0$. At the surface, derivatives of α are continuous, in order to avoid surface currents. The plots below show some numerical equilibria found here, for different values of s, keeping fixed $\lambda = 1.1$. This last value was taken in order to have finite $\beta'(\alpha)$ derivative. Also, $\rho(r) = \rho_c(1 - r^2/R^2)$ and $\chi(\alpha) = k\alpha$ were chosen, and the fixed value $k = 1$ for all plots was taken. Lines correspond to poloidal field lines, whereas the color gradient accounts for toroidal field strength. B_r, B_θ and B_ϕ are the radial, zenithal and azimuthal components of the magnetic field evaluated at the equator, magnetic axis and magnetic axis, respectively.

$s = 10, k = 1$: $E_{tor}/E_{mag} = 0.5\%$



$s = 35, k = 1$: $E_{tor}/E_{mag} = 3\%$



Discussion

All the equilibria found here consist in a mixed poloidal-toroidal field without surface currents and with a dominant poloidal component. For the cases studied so far (s up to 35) he toroidal energy is only a few percent of the magnetic energy even in cases where the toroidal field strength is comparable to the poloidal one; in the latter cases the region where the toroidal field lies is smaller in size. This behavior of the energy has been reported recently by Lander & Jones, perhaps due to the fact that only the first multipole of the magnetic field is considered outside the star (Lander & Jones 2012). Instead, our code allows for an arbitrary number of multipoles, so we expect to examine this issue in a more detail and to check whether it is possible to obtain equilibria with a higher toroidal contribution. Once we obtain a wide range of relevant equilibria with consistent physical choices of the arbitrary magnetic functions, their dynamical stability could be analyzed using either a perturbative analysis or the time-evolution of such configurations, and get a clearer picture about the role of the stable stratification within these objects.

References

Chandrasekhar, S. & Prendergast, K.H., 1956, Proc. Nat. Acad. Sci., 42, 5

Grad, H. & Rubin, H., 1958, *Int. Atom. Ener. Ag. Conf. Proc.* 31 (Genova), p. 190

Haskell, B., et al, 2008, MNRAS, 385, 531

Lander, S.K. & Jones, D.I., 2009, MNRAS, 395, 2162

Lander, S.K. & Jones, D.I., 2012, MNRAS, 424, 482

Reisenegger, A., 2009, A&A, 499, 557

Shafranov, V.D., 1966, in *Reviews of Plasma Physics* (New York: Cons. Bureau), 2, 103

Yoshida, S. &, Eriguchi, Y., 2006, ApJ, 164, 156

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