Determination of QCD phase diagram from the imaginary chemical potential

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This talk presents an overview of our research on the QCD phase diagram.
1. Sign problem of LQCD at real quark chemical potential


Phase factor

\[ e^{2i\theta} = \frac{\det[D(\mu_q)]}{\det[D(\mu_q)]^*} = \frac{(\det[D(\mu_q)])^2}{|\det[D(\mu_q)]|^2} \]

In the two-flavor case, the average of the phase factor is

\[ \langle e^{2i\theta} \rangle = \frac{Z_{1+1}}{Z_{1+1^*}} \]

Fermion determinant

Z in the 1+1 system with \( \mu_u = \mu_d \)

Z in the 1+1* system with \( \mu_u = -\mu_d \)

Prediction of PNJL model

The mean-field approximation

\[ \langle e^{2i\theta} \rangle \approx \frac{\sqrt{\det H_{1+1^*}}}{\sqrt{\det H_{1+1}}} e^{-\beta V(\Omega_{1+1} - \Omega_{1+1^*})}, \]

\[ \langle e^{2i\theta} \rangle = 0.4 \]

\[ \langle e^{2i\theta} \rangle = 0.0 \]
2. Strategy

1. LQCD has the sign problem at real quark chemical potential.
2. Regions with no sign problem
   I) Imaginary quark chemical potential
   II) Isospin chemical potential
3. Our strategy to determine the QCD phase diagram is
   1) construct a reliable effective model in I)-II).
   2) apply the model to the real quark chemical potential region.
3. Mathematical relation between the partition functions at imaginary and real chemical potentials

\[
\theta = \text{Im} \left[ \frac{\mu}{T} \right] \\
Z_{GC}(\theta) = \sum_{n=0}^{\infty} Z_C(n) e^{in\theta} \\
Z_{GC}(\mu) = \sum_{n=0}^{\infty} Z_C(n) e^{n\mu/T}
\]
4. Imaginary chemical potential

Roberge-Weiss Periodicity

\[ Z(\theta + 2\pi/3) = Z(\theta) \quad \theta = \text{Im}(\mu/T) \]
4.1 Derivation of the RW periodicity

\[ Z (\theta) \equiv \text{Tr} \left[ e^{-\beta H + \hat{N} \theta} \right] = \int DqDqD\bar{q} \exp[-S(0)] \]

Boundary condition:
\[ q \left( \frac{1}{T} \right) = -e^{i\theta} q (0) \]

\[ Z (\theta - 2\pi / 3) = \int DqDqD\bar{q} \exp[-S(0)] \]

Boundary condition:
\[ q \left( \frac{1}{T} \right) = -e^{i(\theta - 2\pi / 3)} q (0) \]

Hence,
\[ Z (\theta) = Z (\theta - 2\pi / 3) \]

The RW periodicity is a remnant of the Z3 symmetry in the pure gauge limit.
4.2 The extended $Z_3$ transformation

$Z(\theta)$ is invariant under the combination of the $Z_3$ transformation

$q \rightarrow Uq, \quad A \rightarrow UA U^{-1} - \frac{i}{g}(\partial U)U^{-1}$,

and the transformation for theta

$\theta \rightarrow \theta + \frac{2\pi}{3}$

$Z(\theta) \leftrightarrow Z(\theta + \frac{2\pi}{3})$

This combination is called the extended $Z_3$ transformation.
4.3 Symmetries of QCD

Y. Sakai, K. Kashiwa, H. Kouno and M. Yahiro,

QCD has the extended $Z_3$ symmetry in addition to the chiral symmetry

This is important to construct an effective model.

The Polyakov-loop extended Nambu-Jona-Lasinio (PNJL) model

Fukushima; PLB591
5. Polyakov-loop Nambu-Jona-Lasinio (PNJL) model

Two-flavor

\[ \mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{qrk}} + \mathcal{L}_{\text{glu}} + \mathcal{L}_{\text{g.f}} \]

Fukushima; PLB591

\[ \mathcal{L} = \bar{q}(i\gamma_{\nu}D^{\nu} - m_0)q + G_s[(\bar{q}q)^2 + (\bar{q}\gamma_5\tau q)^2] - \mathcal{U}(\Phi[A], \Phi^*[A], T) \]

Quark part (Nambu-Jona-Lasinio type)

Gluon potential

Rößner, Ratti, Weise; PRD75

\[ \Phi = \frac{1}{3} \text{Tr}_c e^{-iA_4/T} \]

It reproduces the lattice data in the pure gauge limit.
5.1 Model parameters

\[ m_0 = 5.5 \text{ MeV}, \]
\[ G_s = 5.498 \text{ [GeV}^{-2}] \]
\[ \Lambda = 631.5 \text{ MeV}, \]
\[ \int_0^\Lambda d^3 p \]

The thermodynamic potential is calculated by the mean-field approximation.

\[ M_\pi = 138 \text{ MeV} \]
\[ f_\pi = 93.3 \text{ MeV} \]
6. Numerical results of the PNJL model
Phase diagram for deconfinement phase trans.

Lattice data: Wu, Luo, Chen, PRD76(07)

PWJL

RW

Lattice data: Wu, Luo, Chen, PRD76(07)

RW periodicity
Phase diagram for chiral phase transition

\[ \theta = \frac{\mu_I}{T} \]

\[
\begin{align*}
T/T_c & \quad 1.6 \\
\theta/(\pi/3) & \quad 0 \\
\end{align*}
\]
7. Model building

A. PNJL with higher-order quark-quark interactions.


B. PNJL with an effective four-quark vertex depending on Polyakov loop.

Phase diagram for chiral phase transition

Forcrand, Philipsen, NP B642

\[ \theta = \frac{\mu_I}{T} \]

\[ \Theta \text{-even higher-order interaction} \quad \frac{\theta}{(\pi/3)} \]
Deconfinement

Chiral difference

$G_{S8}[(\bar{q}q)^2 + (\bar{q}i\gamma_5\bar{q})^2]^2$

Higher-order correction

8-quark ($\Theta$-even)

Forcrand, Philipsen

$\theta = \mu_1/T$

$T/T_c$

$\theta/(\pi/3)$
Vector-type interaction

8-quark ($\Theta$-even)

$$G_{S8}[(\bar{q}q)^2 + (\bar{q}i\gamma_5\tau q)^2]^2$$

Vector-type ($\Theta$-odd)

$$G_v(\bar{q}\gamma_v q)^2$$

Forcrand, Philipsen N
PB642
Imaginary quark $\mu$

PNJL + 8-quark + vector

$T/T_c$

$\theta/(\pi/3)$

Lattice/Forcrand (02)

Their strength can be determined at imaginary $\mu$.

Imaginary both quark and isospin $\mu$

low T

LQCD

PNJL

Real isospin $\mu$

$\Phi \neq 0$

$\sigma \neq 0$

$\pi \neq 0$

2nd order $\pi$

cross over $\sigma$

cross over $\Phi$

Lattice

T [GeV]

$\mu_1$ [GeV]

1st order

CEP

Deconfined Phase

Confined Phase

Lattice

Kogut, Sinclair (04)

D’Elia, Sanfilippo (09)
No eight-quark interaction

With eight-quark interaction

7.2 Model building

A. PNJL with higher-order interactions


B. PNJL with an effective four-quark vertex depending on Polyakov loop


Entanglement PNJL (EPNJL) model
Model building (B)

B. PNJL with an effective four-quark vertex depending on Polyakov loop

Entanglement interactions such as $G_s(\Phi)\sigma^2$

Extended $Z_3$ symmetry

More precise discussion by K. Kondo, in Hep-th:1005.0314.
Phase Diagram by original PNJL

Phase Diagram by entanglement- PNJL
Entanglement PNJL (EPNJL) model

Isospin chemical potential

Real quark-number chemical potential
8. Short summary

1) We proposed two types of PNJL models. One has higher-order interactions and the other has an entanglement vertex.

2) Both the models give the same quality of agreement with LQCD data.

The question: Which model is better?
The entanglement PNJL may be more reliable.
The order is first order for small and large quark masses, but second order for intermediate masses.

Order of RW phase transition
9. The epoch of the QCD phase transition in the cosmic evolution

Theta-vacuum
in the epoch of the QCD phase transition

\[ \mathcal{L}_{\bar{\theta}} = \bar{\theta} \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu}^a F_{\sigma\rho}^a \]

\[ \Lambda_{QCD} \approx T_c \]

Sphalerons are so activated as to jump over the potential barrier.

\[ \bar{\theta} > 0 \]
Lagrangian with theta-bar dependent anomaly

\[
\mathcal{L} = \bar{q}(i\gamma_\nu D^\nu - m)q - \mathcal{U}(\Phi[A], \Phi[A]^*, T)
+ G_1 \sum_{a=0}^{3} \left[ (\bar{q}\tau_a q)^2 + (\bar{q}i\gamma_5 \tau_a q)^2 \right]
+ 8G_2 \left[ e^{i\theta} \det (\bar{q}_R q_L) + e^{-i\theta} \det (\bar{q}_L q_R) \right],
\]

(a) PNJL

(b) EPNJL

Inhomogeneous BBN
10. Summary

1. For the two-flavor case, we proposed two models.
   A) The PNJL model with high-order quark-quark interactions.
   B) The PNJL model with the effective vertex depending on the Polyakov loop: the EPNJL model
   At the present stage, we think that the EPNJL model is more reliable.

2. The EPNJL model predicts that there is a CEP in the real quark chemical potential region.

3. The EPNJL model also predicts that the chiral and deconfinement transitions are first order even at zero chemical potential, if the theta vacuum has a large value of theta.
QCD partition function \[ Z(\theta) \Rightarrow Z(\theta - 2\pi k / 3) \]

\[ Z_3 \text{ transformation} \]

\[ q \rightarrow Uq, \quad A \rightarrow UAU^{-1} - \frac{i}{g} (\partial U)U^{-1}, \]

where

\[ U(x, \tau) \] is an element of SU(3) with the boundary condition

\[ U(x, 1/T) = e^{i2\pi k/3} U(x, 0) \]

for any integer \( k \)