Phase Structure of QCD
from the Lattice

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We thank the international scientists communities for various supports and encouragements after the earthquake of March 11.
epicenters with $M \geq 3$ in a day (24 hrs).

=> See report by S. Hashimoto @ Lattice 2011 for total damages.
Univ. of Tsukuba

- No personal injuries
- Damages ≥ $90Mi.

We are trying to do our bests to recover the full scientific activities. Your supports have been/are big sources of our power!
To which extent are these pictures confirmed on the lattice?

K. Fukushima and T. Hatsuda
Rep. Prog. Phys. 74, 014001 ('11)
• $\mu = 0$
  * O(4) scaling / Scaling tests on the lattice
  * $T_c$ -- Resolution of the disagreement / Remaining issues

• $\mu \neq 0$
  * Difficulties and tricks
  * Physics / phase structure at $\mu \approx 0$
  * Around the heavy quark limit

✓ Descriptions of works presented at Lattice 2011 are mostly based on my rough memo. Apologize if they are incorrect!
$\mu = 0$
GL effective models + lattice results

Lattice studies with staggered-type quarks
=> Physical point locates in the crossover region

In fixing details of the plot, critical scaling based on universality argument plays an essential role.
QCD transition around the chiral limit

Effective 3d $\sigma$ model with the same flavor-chiral symmetry of massless QCD (continuum)

\[ \mathcal{L} = \text{Tr} \partial_i M^\dagger \partial_i M + \mu^2 \text{Tr} M^\dagger M + \lambda_1 \text{Tr} (M^\dagger M)^2 + \lambda_2 (\text{Tr} M^\dagger M)^2 + c_{U(1)_A} (\text{det} M + \text{det} M^\dagger) \leftrightarrow U(1)_A \text{ anomaly} \]

\[ M_{ab} \sim \left\langle \tilde{q}_a \frac{1 + \gamma_5}{2} q_b \right\rangle \]

- $N_F \geq 3$: 1st order
- $N_F = 2$: depends on the magnitude of the anomaly
  - when anomaly strong: the $\sigma$ model $\approx O(4)$ Heisenberg model
    => 2nd order with established critical properties

\[ M/h^{1/\delta} = f(t/h^{1/\beta \delta}) \]

<table>
<thead>
<tr>
<th>$1/\beta \delta$</th>
<th>0.537(7)</th>
</tr>
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<tbody>
<tr>
<td>$1/\delta$</td>
<td>0.2061(9)</td>
</tr>
</tbody>
</table>

\[ h \sim m_q \quad \Rightarrow \quad \text{chiral violating coupling (external mag. field)} \]
\[ t \sim (T - T_c)/T_c \quad \Rightarrow \quad \text{chiral symmetric coupling (reduced temperature)} \]

- when anomaly negligible around $T_c$
  => fluctuation-induced (weakly) 1st order
In case anomaly negligible around $T_c$,

$N_F = 2$: 1st order chiral trans. with Ising crit. end point

$N_F = 3$: smaller 1st order region <= anomaly was a source of the $M^3$ term

To discriminate the pictures, non-perturbative test on the lattice needed.

$O(4)$ scaling is a powerful guide here.
Flavor-chiral sym. on the lattice

No-go theorem (Nielesen-Ninomiya): one flavor lattice fermion cannot be local and chiral simultaneously.

Chiral symmetry cannot be simply realized on the lattice.

=> several options for the quark action with different flavor-chiral properties:

- Wilson-type / staggered-type / domain-wall / overlap / ...

**Wilson-type quarks:** violate the chiral symmetry at $a > 0$.

Many studies are being made.

**Pros:** ✓ Describes a single flavor. => Flavor symmetry exact.
✓ Continuum limit exists. <= The chiral sym. is restored in the cont. limit.

**Cons:** ❖ Explicit violation of the chiral sym. at $a > 0$.
❖ Light quarks expensive.

**Lattice chiral quarks:** domain-wall / overlap

Still quite expensive to simulate.
Real applications to $T > 0$ have just started! => HotQCD, JLQCD
So far, most large-scale simulations at finite $T$ and $\mu$ have been made with

**Staggered-type quarks**

**Pros:**
- ✓ Relatively cheap to simulate.
- ✓ A modified chiral sym. preserved: $U(1) [\cong O(2)]$ taste-chiral sym.

**Cons:**
- ✗ 4 copies of identical fermions ("tastes") in the cont. lim. for each flavor.
  
  $\Rightarrow$ "4th root trick" to remove unwanted 3: $\det M \Rightarrow [\det M]^{1/4}$
  
  $\Downarrow$

- ✗ Non-local $\Rightarrow$ Universality arguments fragile.
  - $? \text{ continuum limit?}$
    - $\leq$ Empirically OK if the continuum limit is taken first.

  - $? \text{ chiral scaling on finite lattices?}$

- ✗ Taste violation problem at $a>0$ $\Rightarrow$ errors in flavor identifications.
  
  (e.g.) many $\pi$'s in the taste space, one is light due to the taste-chiral sym.
  
  Lightest $\pi$ (pNG $\pi$) usually treated as "physical".
  
  Other $\pi$'s do contribute in dynamical effects.
  
  $\Rightarrow$ lattice artifacts.

$m_q$ is effectively much heavier.
Recently, it was noted that a good control of the taste-violation is essential to obtain physical results with staggered-type quarks.

**Improved staggered quarks**

Various actions proposed to milden lattice artifacts including the taste violation:

- asqtad
- p4
- HYP
- stout
- HISQ
- ...

The extent of improvement differs depending on the action.

\[ \frac{(m_\pi^2 - m_G^2)}{(200 \text{ MeV})^2} \]

**Orginos et al, hep-lat/9909087**

\[ m_\pi/m_\rho = 0.55 \]

**Bazavov-Petreczky**

(HotQCD)

arXiv:1012.1257

The magnitude of the taste violation

- HISQ < stout < asqtad < p4

heavy “π” masses (at \( T \sim 170 \text{MeV} \) with \( m_{\pi^{\text{NG}}} \sim 135 \text{ MeV} \))

- Nt\~8  \sim 400-600
- \sim 300-500
- \sim 200-400

- Nt\~12  \sim 200-350

- asqtad  \<  p4
- stout
- HISQ
- stout

unimproved

asqtad
O(4) scaling tests on the lattice

Wilson-type quarks ($N_f=2$)

Proper renormalization needed to recover the chiral symmetry in the continuum limit.

$$M \sim \langle \bar{\psi} \psi \rangle_{\text{sub}} = 2m_q a Z \sum_x \langle \pi(x) \pi(0) \rangle$$

via axial W.I.  Bochicchio et al. (85)

QCD data vs. O(4) scaling function and exponents

<table>
<thead>
<tr>
<th>Study</th>
<th>Details</th>
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</table>
| Iwasaki et al. (QCDPAX) | PRL78(‘97)  
- Iwasaki gauge + Wilson  
- Nt=4, $m_\pi \sim 600-900$ MeV |
| AliKhan et al. (CP-PACS)| PRD63(‘01)  
- Iwasaki gauge + Clover  
- Nt=4, $m_\pi \sim 600-1000$ MeV |
| Bornyakov et al. (QCDSF) | PRD82(‘10)  
- plaquette gauge + Clover  
- Nt = 8,10,12, $m_\pi \approx 420-1300$ MeV |

No indication of 1st order chiral transition.

QCD data well described by the O(4) scaling function with O(4) exponents.

- Consistent with the O(4) scaling, though quarks are heavy.
Nógrádi (Budapest-Wuppertal) @ Lat11

- Symanzik/tree + 6-level-stout-clover $N_f=2+1$
- fixed scale approach at 6 $\beta$ values

$$M \sim \Delta \bar{\psi} \psi = \langle \bar{\psi} \psi \rangle_T - \langle \bar{\psi} \psi \rangle_{T=0}$$

Comparison with staggered ($\text{stout}, N_t=8-12$) at $m_\pi \approx 545$ MeV, $m_K \approx 612$ MeV

susceptibilities, renormalized Polyakov loop $\Rightarrow$ well consistent with each other.

Continuum thermodynamics feasible with improved Wilson.

No $O(4)$ scaling tests yet.

Burger (tmfT) @ Lat11, 1102.4530

- Symanzik/tree + maximally twisted Wilson $N_f=2$
- $N_t=8-12$ $m_\pi \approx 320-480$ MeV

$O(4)$ fit for $m_\pi \approx 320-480$ MeV works well

$\Rightarrow T_c = 160-270$ MeV.

Difficult to discriminate between $O(4)$ and 1st order yet.
O(4) scaling tests on the lattice

Staggered-type quarks

★ O(4) vs. O(2)

The symmetry of 4-taste staggered quark action is the O(2) taste-chiral symmetry. This is so also with the 4th root trick \( \det M \Rightarrow [\det M]^{1/4} \), therefore,

Sym. of the system in the chiral limit = O(2) for any \( N_F \).

\[ \Rightarrow \] When the chiral transition is 2nd order on the lattice, we expect O(2) scaling not O(4).

O(4) may be realized when (1) continuum extrapolation, and then (2) chiral extrapolation.

In practice, O(2) \( \approx \) O(4) numerically.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>0.349</td>
<td>1.319</td>
<td>4.780</td>
</tr>
<tr>
<td>4</td>
<td>0.380</td>
<td>1.453</td>
<td>4.824</td>
</tr>
</tbody>
</table>

Caveat: Universality may be inapplicable on finite lattices due to the non-locality.

Ejiri et al. (BNL-Bielefeld)
PRD80(09)

- p4, \( N_F=2+1 \)
- \( N_t = 4, m_{ud}/m_s \approx 1/80-1/20 \)
It turned out from intensive studies of $T>0$ QCD with staggered-type quarks around ’09-'11, a good control of taste violation essential to extract physical conclusions with staggered-type quarks.

**Improved staggered quarks ($N_F=2+1$)**

Ejiri et al. (BNL-Bi) PRD80('09) ($N_t = 4$); Lat10 ($N_t = 8$)

- p4, $N_t=4$, $m_s \approx$ physical, $m_l/m_s = 1/80 - 1/20$ ($m_{\pi}^{pNG} \approx 75 - 150$ MeV)

Consistent with $O(2)$

- $m_s \approx$ physical, $m_l/m_s = 1/27 - 1/20$
Improved staggered quarks ($N_F=2+1$)

With staggered-type quarks, we should have taken the cont. limit prior to the chiral scaling studies.

If these properties remain also after taking the cont. limit, ...  

⇒ The phys. point dominated by the $O(N)$ scaling  
⇒ 2nd order chiral transition for $N_F=2$  
⇒ Tricritical point locates lower than $m_{s^{\text{phys}}}$

Consistent with small $m_c$ for $N_F=3$:

$m_{\pi^c} \leq 45$ MeV  
**HISQ**  
Nt=6 Ding et al. @ Lat11

$m_c/m_{ud^{\text{phys}}} \leq 0.12$  
**stout**  
Nt=6 Endrodi et al. @ Lat07

Consistent with broken $U(1)_A$ at $T_\approx T_c$ and above:

**HISQ, DW**  
$N_F=2+1$  
HotQCD @ Lat11

**Overlap**  
$N_F=2$  
Cossu (JLQCD) @ Lat11
\( T_c \) at the physical point

\( T_c \) at the phys. pt. in the cont. limit: available so far with stag. quarks only.

\( T_c \) \( \leq \) susceptibility peak (chiral / quark# / Polyakov loop / ...) etc.
+ scale determined by \( T=0 \) simulations

Crossover \( \Rightarrow \) uncertainty in the value of \( T_c \) depending on the definition.

There was a big discrepancy about the value of \( T_c \) beyond the uncertainty ('06-'09).
\( \Rightarrow \) major discrepancy resolved in '10-'11.
\( \leq \) importance of controlling the taste violation

(see KK @ Lat10)
Latest values of $T_c$ with improved staggered quarks ($N_f=2+1$) at the physical point and in the continuum limit

**Stout**  Borányi et al. (Wuppertal-Budapest) JHEP 09 ('10)

<table>
<thead>
<tr>
<th></th>
<th>$\chi_\psi/T^4$</th>
<th>$\Delta_{l,s}$</th>
<th>$\langle \bar{\psi}\psi \rangle_R$</th>
<th>$\chi_2/T^2$</th>
<th>$\epsilon/T^4$</th>
<th>$(\epsilon - 3p)/T^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_c$</td>
<td>147(2)(3)</td>
<td>157(3)(3)</td>
<td>155(3)(3)</td>
<td>165(5)(3)</td>
<td>157(4)(3)</td>
<td>154(4)(3)</td>
</tr>
</tbody>
</table>

Identified with “inflection point”

**HISQ + asqtad**  HotQCD @ QM 11, Lat 11

$T_c = 157(4)(3)(1)$ MeV

Preliminary

with $\chi_{\text{chiral}}$

(similar with Polyakov suscept.)

Major discrepancy in $T_c$ removed. <= good control of the taste violation important

Remaining disagreements: shape of $\chi$, EOS, operator dependence in $T_c$, etc.
$T_c$ under external magnetic field

Motivated by the chiral magnetic effect, phase structure under uniform background magnetic field is studied.

D’Elia-Mukherjee-Sanfilippo, 1005.5365

- KS $N_F=2$ => $T_c$ higher, trans. sharper
- $N_t=4$ $m_{\pi}^{pNG} \approx 195\text{-}480$ MeV with B.

Endrődi (Wuppertal-Budapest) @ Lat11

- stout $N_F=2+1$ => $T_c$ lower with B.
- $N_t=8\text{-}12$ $m_{\pi} \approx 135$ MeV
  Remains crossover up to $eB \geq 1$ GeV$^2$.
  D’Elia’s result recovered at large $m_q$.

T. Ishikawa (HotQCD) @ Lat11

- DW $N_F=2$ => $L_s \geq 94$ needed to study low modes which are relevant to the CME.
\[ \mu \neq 0 \]
Difficulties of the lattice at $\mu \neq 0$

- **LQCD at $\mu \neq 0$**
  \[
  U_4 \rightarrow \begin{cases} 
  U_4 e^{\mu a} \\
  U_4 e^{-\mu a}
  \end{cases} \quad \cdots \quad \text{positive } t \text{ direction}
  \\
  \begin{cases} 
  \text{nagative } t \text{ direction}
  \end{cases}
  \]

- **Complex phase problem (sign problem)**
  \[
  [\det M(\mu)]^* = \det M(-\mu^*)
  \]
  => Exponential cancellation due to the phase fluctuation of $\det M$

- **Techniques for small $\mu/T$**
  - Taylor expansion
  - Multi-parameter reweighting
  - Canonical ensemble
  - Analytic continuation from imaginary $\mu$
  - Complex Langevin
  - **Combination of them**
    \[
    [ \text{+ density of state method / cumulant expansion / ... } ]
    \]

- $\Rightarrow \mu/T \leq O(1)$ accessible
Physics at $\mu/T \leq O(1)$

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n$$

$$c_n(T) = \frac{1}{n!} \frac{N_t^3}{N_s^3} \frac{\partial^n \ln Z}{\partial (\mu_q/T)^n} \bigg|_{\mu_q=0}$$

$$\Delta p = p(\mu) - p(0)$$

Allton et al. (Bielefeld-Swansea)
PRD 71 ('05)
$p4, N_f=2, m_q/T=0.4, n \leq 6$

Ejiri et al. (WHOT-QCD)
PRD 82 ('10)
Clover, $N_f=2, m_{PS/m_V}=0.65, n \leq 2$

thru measurement of quark propagators

$$D_1 = N_f \text{tr} \left( M^{-1} \frac{\partial M}{\partial \mu} \right)$$
$$D_2 = N_f \left[ \text{tr} \left( M^{-1} \frac{\partial^2 M}{\partial \mu^2} \right) - \text{tr} \left( M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} \right) \right]$$

etc.
\[ \frac{n_f}{T^3} = \frac{1}{VT^3} \frac{\partial \ln Z}{\partial (\mu_f/T)} = \frac{\partial (p/T^4)}{\partial (\mu_f/T)} \]

quark number susceptibility

isospin susceptibility

charge fluctuation

\[ \frac{\chi_q}{T^2} = \left( \frac{\partial}{\partial (\mu_u/T)} + \frac{\partial}{\partial (\mu_d/T)} \right) \frac{n_u + n_d}{T^3} \]

\[ \frac{\chi_I}{T^2} = \left( \frac{\partial}{\partial (\mu_u/T)} - \frac{\partial}{\partial (\mu_d/T)} \right) \frac{n_u - n_d}{T^3} \]

\[ \chi_c = \left( \frac{2}{3} \frac{\partial}{\partial \mu_u} - \frac{1}{3} \frac{\partial}{\partial \mu_d} \right) \left( \frac{2}{3} \frac{\partial p}{\partial \mu_u} - \frac{1}{3} \frac{\partial p}{\partial \mu_d} \right) \]

Allton et al. (Bielefeld-Swansea)
PRD 71 ('05)
p4, \( N_f=2, m_q/T=0.4, n \leq 6 \)

Ejiri et al. (WHOT-QCD)
PRD 82 ('10)
\textbf{Clover}, \( N_f=2, m_{PS}/m_V=0.65, n \leq 2 \)

etc.
Generalized/non-linear susceptibilities:

\[ \chi_q^{(n)} = \frac{\partial^n [p(T, \vec{\mu})/T^4]}{\partial (\mu_q/T)^n} \]

\begin{align*}
\text{variance} & \quad \sigma_q^2 = \left\langle (\delta N_q)^2 \right\rangle = VT^3 \chi_q^{(2)} \\
\text{skewness} & \quad S_q = \frac{\left\langle (\delta N_q)^3 \right\rangle}{\sigma_q^3} = VT^3 \chi_q^{(3)} / \sigma_q^3 \\
\text{kurtosis} & \quad \kappa_q = \frac{\left\langle (\delta N_q)^4 \right\rangle}{\sigma_q^4} - 3 = VT^3 \chi_q^{(4)} / \sigma_q^4
\end{align*}

\[ S_q \sigma_q = \frac{\chi_q^{(3)}}{\chi_q^{(2)}}, \quad \kappa_q \sigma_q^2 = \frac{\chi_q^{(4)}}{\chi_q^{(2)}} \]

V-independent combinations:

Comparison with experiment on the chemical freeze-out line

Event-by-event proton distribution by \textit{STAR}

Mumbai: \( N_f=2 \ KS, \ N_t=4,6, M_{\pi^{pNG}} \approx 230 \text{MeV} \)
BNL-Bi: \( N_f=2+1 \ p4, \ N_t=8, M_{\pi^{pNG}} \approx 160 \text{MeV} \)

Lattice artifacts yet to be studied:
* cutoff effects / taste violation
* scale setting error / heavy \( M_\pi \)
* finite \( V \) effects
* extrapolation to the freeze-out pt. etc.

Phase structure at small $\mu/T$
Transition/crossover curve at $\mu \approx 0$

Taylor expansion

$$T_c(\mu_B^2) = T_c \left(1 - \kappa \cdot \frac{\mu_B^2}{T_c^2}\right)$$

Endröli et al., JHEP1104

stout, $N_f=2+1$, $N_t=6-10$, Physical pt., continuum limit.

reflection point:

$$\kappa \left(\frac{x_s}{T^2}\right) = 0.0089(14)$$

$$\kappa \left(\frac{\bar{\psi}\psi_r}{T}\right) = 0.0066(20)$$
**Transition/crossover curve at $\mu \approx 0$**

$O(4)$ scaling around the physical point at $\mu = 0$.

$$M/h^{1/8} = f(t/h^{1/8})$$

$h \sim m_q a \quad \leqslant \text{chiral violating terms}$

$t \sim \beta - \beta_c \quad \leqslant \text{chiral symmetric terms}$

$O(4)$ scaling at small $\mu$

Leading contribution from $\mu$ is $\mu^2$ and chiral symmetric.

$$t = \beta - \beta_{ct} + \frac{c}{2} \left( \frac{\mu_q}{T} \right)^2$$

$c \Rightarrow \text{curvature of the crossover line in the (}\beta, \mu\text{) plane}$

=> We can extract $\kappa$ by an $O(4)$ scaling fit too.

---

Kaczmarek et al. (BNL-Bi) PRD 83, 014504 ('11)

$p4, N_F=2+1 \quad O(2)$ fit

$$\kappa_q = 0.059(2)(4)$$

=>$\kappa = 0.0066(5)$

---

Ejiri et al. (WHOT-QCD) PoS(Lat10)181

**clover, $N_F=2 \quad O(4)$ fit**

$c \approx 0.02--0.05$ (preliminary)

=>$\kappa \sim 0.008$
Imaginary \( \mu \)

No sign problem \( \Rightarrow \) can simulate at \( \mu^2 < 0 \)

\[
\frac{T_{pc}}{T^0_{pc}} = \sum_n d_n \left( \frac{\mu I}{T_{pc}} \right)^{2n} \quad \mu I^2 = -\mu^2
\]

Analytic continuation:
fit \( \mu^2 < 0 \) \( \Rightarrow \) extrapolate to \( \mu^2 > 0 \)

A recent example:
Nagata-A.Nakamura
PRD83('11)
clover, \( N_f = 2 \), \( M_{ps}/M_v = 0.8 \)

\[
\kappa = \left( \frac{d_1}{d_0} \right)^2 \leq 0.006
\]

Model studies: extrapolation works to some extent.

Kim et al., PoS(Lat05)
Potts model

Cea et al., PRD80('09)
KS, \( N_f = 2 \), isospin \( \mu \)

But many precise data points are required.

\( \Rightarrow \) The sign and the magnitude of the curvatures roughly consistent:
\( \kappa \approx 0.005 \text{ -- } 0.009 \) in contrast to \( \kappa \approx 0.02 \) for the freeze-out.

\[
\kappa = \left( \frac{d_1}{d_0} \right)^2 \leq 0.006
\]
Critical point

- **Taylor expansion method:**
  - (1) Gavai & Gupta [PRD78 (2008)], standard staggered $N_f = 2$, $N_t = 6$, volumes up to $V = 24^3$, $m_\pi/m_\rho \approx 0.3$.
  - (2) Schmidt [arXiv:1007.5164], p4 $N_f = 2+1$, $N_t = 4$, volumes up to $V = 24^3$, $m_\pi/m_\rho \approx 0.3$.

- **Reweighting:** Fodor & Katz [JHEP 0404 (2004)], staggered $N_f = 2+1$, $N_t = 4$, volumes up to $V = 12^3$, $m_\pi/m_\rho = \text{physical}$.

- **Canonical approach:** Ejiri et al. [PRD78 (2008)], p4 $N_f = 2$, $N_t = 4$, volume $V = 6^3$, $m_\pi/m_\rho \approx 0.7$.

- **Direct canonical simulation:** New result A. Li et al. [arXiv:1103.3045], clover $N_f = 3$, $N_t = 4$, $V = 6^3$, $m_\pi = 700 - 800$ MeV.
Radius of convergence of the Taylor series

\[ \frac{p}{T^4} = \frac{1}{VT^3} \ln Z = \sum_{n=0}^{\infty} c_n(T) \left( \frac{\mu_q}{T} \right)^n \]

\[ \frac{\chi_q(\mu/T)}{T^2} = \sum_{n=2}^{\infty} \frac{n(n-1)c_n(T)}{n-2} \left( \frac{\mu}{T} \right)^{n-2} = c_{n-2}^x(T) \]

\( \leq \) if the singularity locates on the real axis

\( \text{e.g., if } c_n > 0 \text{ after some finite } n. \)
Radius of convergence of the Taylor series

\[ \frac{p}{T^4} = \frac{1}{VT^3} \ln Z = \sum_{n=0}^{\infty} c_n(T) \left( \frac{\mu_q}{T} \right)^n \]

\[ \frac{\chi_q(\mu/T)}{T^2} = \sum_{n=2}^{\infty} \frac{n(n-1)c_n(T)}{\mu/T} \left( \frac{\mu}{T} \right)^{n-2} = c_{n-2}^x(T) \]

<= if the singularity locates on the real axis
e.g., if \( c_n > 0 \) after some finite \( n \).

=> lower bound for the distance to the crit. pt.

But,

✦ only quite short series available (at most 8, but models often require more)
✦ various estimators for the radius (different at small \( n \))
✦ various quantities for the series (\( \chi \) better than \( p \)? Karsch et al. PLB698(’11))
✦ alternative estimators using, e.g., Padé series etc.

In addition to these ambiguities:

✦ control of lattice artifacts /taste violation
✦ continuum extrapolation
✦ physical point extrapolation

=> Syst. errors not well controlled yet.
Canonical ensemble approach

\[ Z_{GC}(T, \mu_q) = \int DU(\det M(\mu_q/T))^{N_f} e^{-S_g} \]
\[ = \sum_N Z_c(T, N)e^{N\mu_q/T}, \]
\[ \partial(\ln Z_c)/\partial N(T, N) \equiv -\mu_q^*/T \]

Many trials since decades.

Ejiri, PRD 78 (’08)
p4, \(N_f=2, 16^3\times4, m_q/T=0.4\)
1st order for \(\mu_q/T > \approx 2.4\)

Li et al. (χQCD)  arXiv:1103.3045
Clover, \(N_f=3, 6^3\times4, m_\pi\sim\text{700-800MeV}\)

\[ T_E/T_c=0.927(5) \]  
\[ \mu_B/T_c=2.60(8) \]

\[ T/T_c=0.90(1) \]  
\[ T/T_c=0.87(1) \]  
\[ T/T_c=0.85(1) \]

\[ n_B \]

\[ n_B \]

\[ n_B \]

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A histogram method
(a density of state method)

★ \( \mu = 0 \) around the heavy quark limit

H. Saito et al. (WHOT-QCD), arXiv:1106.0974

Effective potential for the plaquette distribution
standard plaquette gauge action + Wilson quark action

\[
V_{\text{eff}}(P, \beta, \kappa) = -\ln w(P, \beta, \kappa)
\]

Reweighting in \( \beta \)

\[
V_{\text{eff}}(P, \beta, \kappa) = V_{\text{eff}}(P, \beta_0, \kappa) - 6(\beta - \beta_0)N_{\text{site}}P.
\]

Reweighting at different \( \beta \) to obtain accurate \( V_{\text{eff}} \) in a wide range of \( P \).

Reweighting in \( \kappa \sim 1/m_q \) [with the hopping parameter expansion]

\[
\Rightarrow \text{phase structure from the shape of } V_{\text{eff}}\]
**μ ≠ 0**

Reweighting factor in κ and μ with the hopping parameter expansion

\[
\ln \left[ \frac{\det M(\kappa, \mu)}{\det M(0, 0)} \right] = 288N_{\text{site}}\kappa^4P + aN_s^3(\Omega_R + ib\Omega_I)
\]

\[a = 3 \times 2^{N_f+2}\kappa^{N_f}\cosh \left(\frac{\mu}{T}\right), \quad b = \tanh \left(\frac{\mu}{T}\right).\]

\[V_{\text{eff}}\] for \(P\) and the Polyakov loop \(\Omega\)

\[\Rightarrow\] phase structure at \(\mu \neq 0\) from the shape of \(V_{\text{eff}}\)

Lines of \(dV_{\text{eff}}/dP = 0\) and \(dV_{\text{eff}}/d\Omega_R = 0\)

- typical double well
- typical single well

\[\lambda = 1.0 \times 10^{-5}\]

\[\lambda = 5.0 \times 10^{-5}\]

\[\lambda = 1.0 \times 10^{-4}\]

\[\beta = 5.69, \quad K = \mu = 0\]

\[K_{\text{cp}}\] at small \(\mu\)

\(K_{\text{cp}}\) for isospin chem. pot.

(leading order hopping param. expansion)

\[\kappa_{\text{cp}}(\mu) = \frac{\kappa_{\text{cp}}(0)}{\cosh \left(\frac{\mu}{T}\right)}\]
Application to the light quark region

\[ V_{\text{eff}}(P, F) = N_F \ln \left| \frac{\det M(\mu)}{\det M(0)} \right| \]

For reweighting with dynamical quarks, \(|\det M|\) is important, in place of the \(\Omega\) for the heavy quark case.

Simulation with \(|\det M(\mu)|\) => reweight to \(\det M(\mu)\)

Sign problem in the phase average <= cumulant expansion + Gaussian approx.

The method looks feasible. Investigation under way.
Imaginary $\mu$

- Similar to the case of $T_c(\mu)/T_c(0)$

$\mu$

must turn back to have the crit. point at $m_q^{\text{phys}}$.
Or, more complicated phase structure at finite $\mu$?

Caveat: unimproved staggered quarks on $N_t=4$ have large lattice artifacts.

dej Forcrand-Philipsen, JHEP0701; 0811
KS Nt=4

dej Forcrand et al.@Lat11
KS Nt=4
Summary
To which extent are these pictures confirmed on the lattice?

K. Fukushima and T. Hatsuda
Rep. Prog. Phys. 74, 014001 (’11)
QCD PHASE STRUCTURE

at $\mu \approx 0$

“Columbia plot”

O(4) probable

K. Fukushima and T. Hatsuda
Rep. Prog. Phys. 74, 014001 (’11)

at $m_q$'s $\approx$ physical values

$\mu/T \leq O(1)$
Qualitative understanding on the phase structure has much progressed.

Tuning to the physical point / extrapolation to the cont. limit require quantitative precisions.

Currently, the staggered-type quarks are the closest to the goal. Here, A good control of taste violation essential to extract physical conclusions with staggered-type quarks.

Studies of EOS, fluctuations, etc. with improved staggered quarks underway.

But, because the fundamental uneasiness with staggered-type quarks, Need to confirm the results with theoretically sound lattice quarks (Wilson-Clover, DW, Overlap).

=> Scaling studies, EOS, .... underway (many developments reported @ Lat10-11)

Besides, many new ideas, steady/new progresses, ... appeared. => The next reviewer will be busy too!.
thank you!